Alhazen's Summation Formulas

Alhazen was physicist as well as a mathematician. His work was grounded in the Greek geometrical tradition, but he also sought after practical empirical results in optics and astronomy. Greek mathematics (with the exception of Diophantus and Heron) does not mention any powers higher than three because they could not be directly interpreted in geometry, i.e. as lengths, areas, or volumes. Alhazen wanted to calculate new results concerning areas and volumes which involved the summation of powers higher than three. For example he computed the volume generated by rotating a parabola about a line perpendicular to its axis of symmetry (see Figure 1). For a modern student this would involve integrating a fourth power polynomial. Alhazen stated that the volume is 8/15 of the volume of the circumscribed cylinder. In the tradition of Eudoxus, he set up upper and lower sums of cylindrical slices, and then let the slices get finer and finer. Since the radius of each cylindrical slice follows a square function, the areas follow a fourth power. The sum of the areas of these slices involves summing fourth powers (Edwards, 1979, p.85).

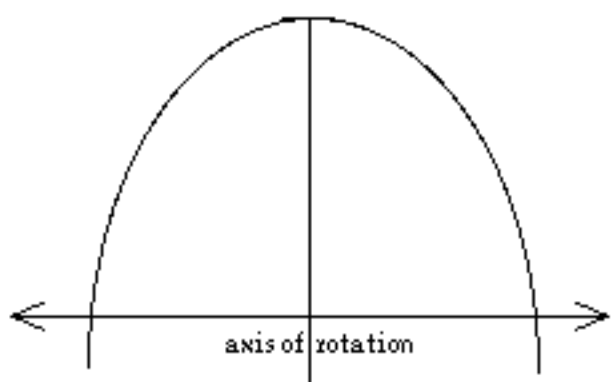

Figure 1

Alhazen derived formulas for the sums of higher powers by coordinating a geometrical interpretation with a numerical representation. His use of areas to represent third and fourth powers broke with the strict geometrical interpretations in Greek mathematics. His method can be used to find a formula for the sum of the powers of the first n integers for any positive integer power. His extension of the concept of powers made sense arithmetically, but was validated through his derivations of new geometric results.

Alhazen derives his formulas by first laying out a sequence of rectangles whose areas represent the terms of the sum (Edwards, 1979, p.84). A rectangle of area a^k is formed using sides of length a^{k-1} and a. He then fills in the rectangle with a series of interlocking strips (see Figures 1, 2 and 3, which are to scale). Alhazen then sets the product of the dimensions of the rectangle equal to the sum of its rectangular parts. Each of the formulas can then be derived from the previous ones. The strips on top, however, involve a double summation in all but the first derivation.

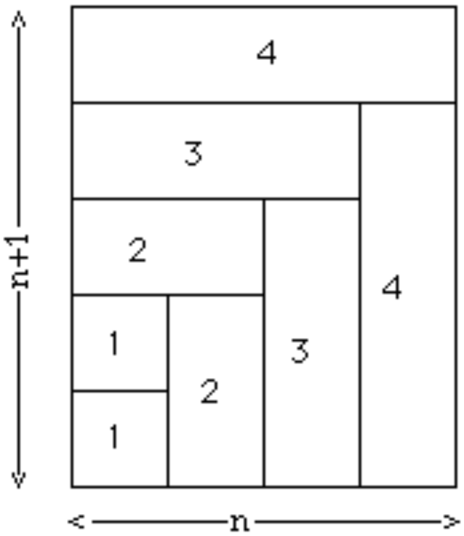

Figure 2

To obtain a formula for sum of the first n integers, see Figure 2.

$$n\,(n+1) = \sum_{i=1}^{n} i + \sum_{i=1}^{n} i$$

(1) $$\frac{1}{2}\, n\,(n+1) = \frac{1}{2}\, n^2 + \frac{1}{2}\, n = \sum_{i=1}^{n} i = 1+2+3+ ....+n$$

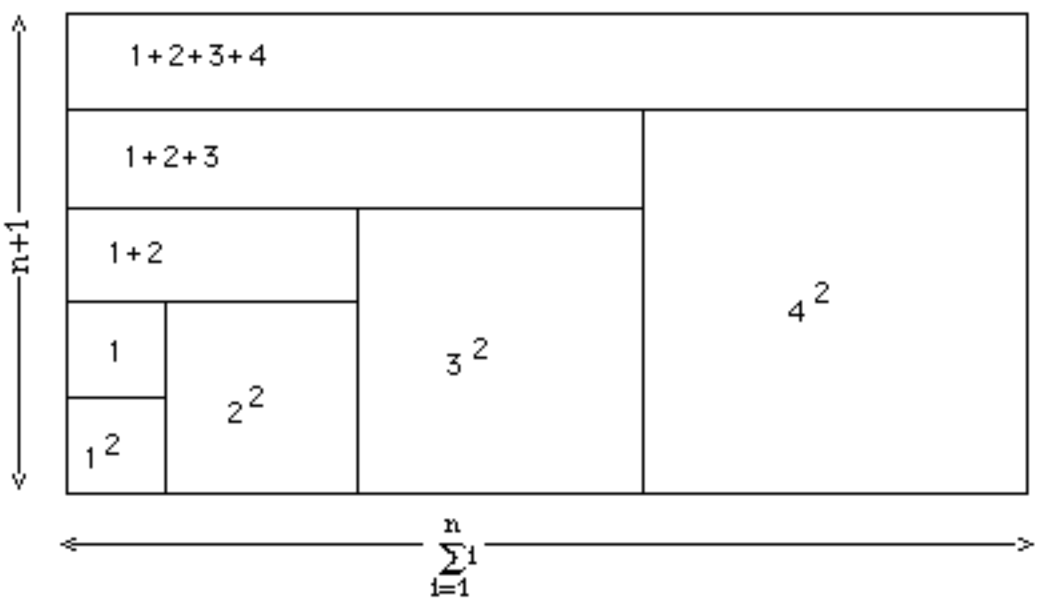

Figure 3

To obtain a formula for the sum of the squares of the first n integers, see Figure 3.

$$\left(\sum_{i=1}^{n} i\right)(n+1) = \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} \left\{ \sum_{k=1}^{i} k \right\}$$

$$\left(\frac{1}{2} n^2 + \frac{1}{2} n\right)(n+1) = \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} \left\{\frac{1}{2} i^2 + \frac{1}{2} i\right\}$$

$$\frac{1}{2} n^3 + n^2 + \frac{1}{2} n = \sum_{i=1}^{n} i^2 + \frac{1}{2} \sum_{i=1}^{n} i^2 + \frac{1}{2}\left(\frac{1}{2} n^2 + \frac{1}{2} n\right)$$

$$\frac{1}{2} n^3 + \frac{3}{4} n^2 + \frac{1}{4} n = \frac{3}{2} \sum_{i=1}^{n} i^2$$

(2) $$\frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n = \sum_{i=1}^{n} i^2 = 1^2+2^2+3^2+ \text{..........} + n^2$$

Note that (1) was used twice in obtaining (2).

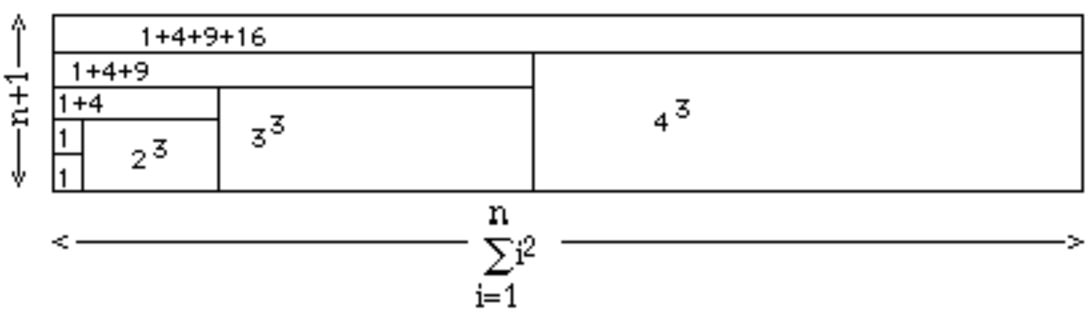

Figure 4

To obtain a formula for the sum of the cubes of the first n integers, see Figure 4.

$$\left(\sum_{i=1}^{n} i^2\right)(n+1) = \sum_{i=1}^{n} i^3 + \sum_{i=1}^{n} \left\{ \sum_{k=1}^{i} k^2 \right\}$$

$$\left(\frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n\right)(n+1) = \sum_{i=1}^{n} i^3 + \sum_{i=1}^{n} \left\{ \frac{1}{3} i^3 + \frac{1}{2} i^2 + \frac{1}{6} i \right\}$$

$$\frac{1}{3}n^4+\frac{5}{6}n^3+\frac{4}{6}n^2+\frac{1}{6}n = \sum_{i=1}^{n} i^3 + \frac{1}{3}\sum_{i=1}^{n} i^3 + \frac{1}{2}\left(\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n\right) + \frac{1}{6}\left(\frac{1}{2}n^2+\frac{1}{2}n\right)$$

$$\frac{1}{3}n^4 + \frac{4}{6}n^3 + \frac{4}{12}n^2 = \frac{4}{3}\sum_{i=1}^{n} i^3$$

(3) $$\frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 = \sum_{i=1}^{n} i^3 = 1^3+2^3+3^3+ \text{..........}+n^3$$

Note that here (2) was used twice and (1) was used once.

A formula for the sum of the first n fourth powers can be derived by continuing this method. First lay out a series of rectangles whose horizontal sides are the cubes and whose vertical sides are once again successive integers. Filling in the strips and proceeding as before will yield the desired formula, although the algebra becomes more tedious. All three of the previous formulas must be used. This derivation is left as an exercise for the reader. This method can be extended to yield a formula for the sum of the first n powers for any integer power. It is a general recursion scheme for these formulas, but at each stage not just one but many of the previous formulas must be used. For future reference here are Alhazen's Formulas:

(1) $1+2+3+ \text{.......}+n=\frac{1}{2}n^2 + \frac{1}{2}n$

(2) $1^2+2^2+3^2+ \text{..........}+n^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$

(3) $1^3+2^3+3^3+ \text{..........}+n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$

(4) $1^4+2^4+3^4+ \text{...........}+n^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$

In order to calculate the volume of the rotated parabola in Figure 1, the actual summation needed is $\sum_{i=1}^{n}(n^2-i^2)^2 = \frac{8}{15}n^5 - \frac{1}{2}n^4 - \frac{1}{30}n$, which can be derived from (2) and (4) by substituting and collecting terms. The formula inside the summation is the square of the radius of each slice (Edwards, 1979, p.84).

It should be stressed here that results about areas and volumes were not given as formulas but always as ratios. This is true about nearly all such mathematical results until the end of the seventeenth century. For example the area of a triangle is one half of the area of the parallelogram that contains it. The volume of a pyramid is one third of the box that contains it.

The area of a piece of a parabola is two thirds of the rectangle that contains it. The area of the circle is $\pi/4$ of the square that contains it. These examples were all known by the second century BC. Alhazen's ratio of 8/15 is a continuation of that tradition, but he had to coordinate geometry and arithmetic in a new way to find it. In the work of John Wallis we will see an elaborate consideration of area ratios used to justify his definition of fractional and negative exponents. He made extensive use of these summations, and in the final validation of his ideas, Alhazen's ratio of 8/15 appeared in one of his tables.