Geometric Constructions

MATHEMATICAL INTENTIONS

Long before Euclid, one of the purposes of geometry was to measure and construct precise shapes and sizes for architecture, agriculture, and decorative arts. Euclid transformed these activities into a field of mathematical inquiry. Introducing axioms, theorems, and proofs makes a construction problem into a logic game with rules.

The geometry of Euclid allows constructions with only two tools: straightedge (a ruler with no marks) and compass (tool for drawing circles from a center point and radius measurement.) The original text of Euclid did not have any pictures. Probably this was because, while scribes could copy text fairly well without errors, they could not be trusted to make an accurate diagram. So instructions for making your own diagrams are included.

A straightedge need not be a ruler. A tightly stretched string, rubbed with chalk, will produce a straight line on a chalkboard or sidewalk. The same string, held down at one end, will allow you to draw a circle.

The rules on what is allowed in a construction are given by Euclid's first three postulates (axioms).

Postulate 1. To draw a straight line from any point to any point.

Postulate 2. To produce a finite straight line continuously in a straight line.

Postulate 3. To describe a circle with any center and radius.

For example, Euclid's first proposition is instructions for constructing an equilateral triangle with a given line segment as a line (the first paragraph in Proposition 1, below), followed by a proof that this is indeed an equilateral triangle.

(From Euclid, via David Joyce's website)

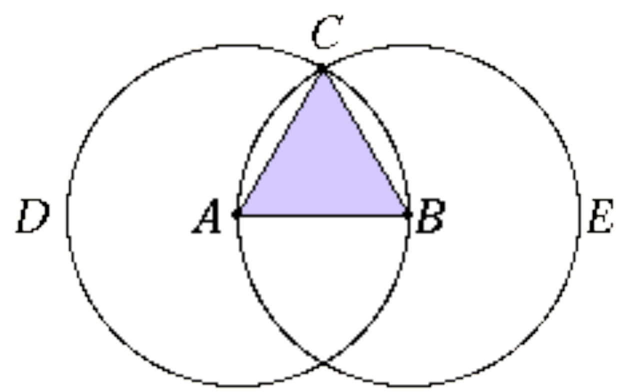
Proposition 1

To construct an equilateral triangle on a given finite straight line.

Let AB be the given finite straight line.

It is required to construct an equilateral triangle on the straight line AB.

Describe the circle BCD with center A and radius AB. Again describe the circle ACE with center B and radius BA. Join the straight lines CA and CB from the point C at which the



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> circles cut one another to the points A and B.
>
> Now, since the point A is the center of the circle CDB, therefore AC equals AB. Again, since the point B is the center of the circle CAE, therefore BC equals BA.
>
> But AC was proved equal to AB, therefore each of the straight lines AC and BC equals AB.
>
> And things which equal the same thing also equal one another, therefore AC also equals BC.
>
> Therefore the three straight lines AC, AB, and BC equal one another.
>
> Therefore the triangle *ABC* is equilateral, and it has been constructed on the given finite straight line *AB.*

Some of the first propositions in Euclid prove that you can do certain basic constructions: construct a line perpendicular to another line, through any point on or off the line. In essence, this allows you to use a right-angle tool such as a T-square or drafting triangle, which makes a perpendicular in one step, rather than the 3 steps needed with straightedge and compass.

Construction problems were popular with ancient mathematicians. Two that remained unsolved for millennia were trisecting the angle (find a general method using straightedge and compass that will construct an angle with 1/3 the measure of *any* given angle) and squaring the circle (use only straightedge and compass to construct a square that has the same area as a given circle.) These were finally shown to be impossible in the 1800s using advanced algebraic concepts: Galois theory and the theory of equations.

In the middle ages, Islamic mathematicians expanded on Euclid's tools. There were many devices that drew curves. See (Dennis, 1995). Some of their work involved constructing conic sections such as parabolas, and intersecting them to solve higher degree equations.

By Descartes' day, linkages to draw curves were a prominent mathematical idea, as were other physical methods, such as the curve traced by a point on the rim of a rolling wheel (a cycloid.) A curve traced by a moving point is called a *locus*; this idea has been brought back by dynamic geometry software. Descartes introduced the idea of describing such curves with coordinates and equations. Euler, 100 years later, completed the job of algebraizing curves; since then, we have thought of curves as defined by equations, not the other way around.

Geometric constructions are currently a topic of mathematical recreation. (Martin, 1997) discusses variations on straightedge and compass (compass only, straightedge with two marks, etc.) and proves what can be done with each set of tools.

Mathematical origami is a subject of current interest. One of the themes is trying to find a set of axioms for basic paper folding moves, in a way that accurately models what happens with actual paper. (Lang, 2009) is a collection of articles from a conference on mathematical origami.

Try this (1). Learn how to do these basic constructions with straightedge and compass only.

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a) Given a line segment, a line, and a point P that is on the line, construct a segment along the line, starting at P, that has the same length as the segment. (That is, copy a length.)
b) Given a point *not* on a line, construct the perpendicular to the line through the point.
c) Given a point on a line, construct the perpendicular to the line through the point.
d) Given a point not on a line, construct the parallel to the line through the point.
e) Given a line segment, construct its perpendicular bisector.
f) Given an angle, a different line *l*, and a point P on the line, construct an angle with vertex P, *l* as one side, and having the same measure as the first angle. (That is, copy an angle.)
g) Bisect a given angle.
h) Given a line segment, construct an equilateral triangle with the segment as a side.
i) Given a line segment, construct a square with the segment as a side.
j) Given three line segments, construct a triangle with the lengths of those segments as side lengths. (SSS congruence criterion.)
k) Given two segments and an angle, construct a triangle with those two sides and the given angle as the included angle. (SAS congruence criterion.)
l) Given two angles and a segment, construct a triangle with those two angles and the given segment as the included side. (ASA congruence criterion.)
m) Given two segments and an angle, construct all possible triangles with those two sides and the given angle as a non-included angle. There may be more than one, depending on the lengths and angles involved. (SSA; not a congruence criterion.)
n) Given two angles construct a triangle with those two angles. (AA similarity criterion.)

Try this (2). Learn how to do the constructions above with a dynamic geometry program.

Note that, in these programs, constructing is different from drawing. There is a tool to construct/draw a line segment. If you simply draw a perfect-looking square with the line segment tool, it will become distorted if you drag any of the endpoints. This is known as the *drag test*: if you drag any part of the construction, it should retain all the qualities you were intending to construct.

The *generality test:* Your object should also have no additional qualities. For example, if you were intending to construct a rectangle, it should be draggable to the shape and size of any possible rectangle, not just a rectangle with base 1 and height 2, or a scalable rectangle whose height is always twice its base.

If you follow the straightedge and compass procedures you developed, the construction should pass both the drag test and the generality test.

Try this (3). (Beyond the basics.)

a) Learn how to construct the golden ratio, and a general regular pentagon.
b) Do Archimedes's construction of inscribed and circumscribed polygons, starting with a hexagon, of a circle, and do the calculations of perimeters to get an approximation for pi. (Requires some trigonometry.)

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c) Look up some axioms for origami, and do the constructions listed in the first Try this exercises. What are some constructions that you can do with origami that you can't do with straightedge and compass? In particular, learn to trisect a general angle.

References

Dennis, D. (1995). Historical Perspectives for the Reform of Mathematics Curriculum: Geometric Curve Drawing Devices and their Role in the Transition to an Algebraic Description of Functions. Ph.D. dissertation, Cornell University. http://www.quadrivium.info/mathhistory/CurveDrawingDevices.pdf

Euclid. *Euclid's Elements* (Joyce, David E., Tran.). http://aleph0.clarku.edu/~djoyce/java/elements/elements.html

Lang, R. J. (2009). *Origami 4*. A K Peters Ltd.

Martin, G. E. (1997). *Geometric Constructions.* Springer.

Dynamic geometry software imitates straightedge and compass constructions. Those listed below also will construct curves from equations.

GeoGebra. http://www.geogebra.org

Geometer's Sketchpad. http://www.keypress.com

Cabri. http://www.cabri.com

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