The Cycloid: Tangents, Velocity Vector, Area, and Arc Length

[This is Chapter 2, section 13 of *Historical Perspectives for the Reform of Mathematics Curriculum: Geometric Curve Drawing Devices and their Role in the Transition to an Algebraic Description of Functions*; http://www.quadrivium.info/mathhistory/CurveDrawingDevices.pdf Interactive applets for the figures can also be found at Mathematical Intentions.]

The circle is the curve with which we all have the most experience. It is an ancient symbol and a cultural icon in most human societies. It is also the one curve whose area, tangents, and arclengths are discussed in our mathematics curriculum without the use of calculus, and indeed long before students approach calculus. This discussion can take place, because most people have a lot of experience with circles, and know several ways to generate them. Pascal thought that, second only to the circle, the curve that he saw most in daily life was the cycloid (Bishop, 1936). Perhaps the large and slowly moving carriage wheels of the seventeenth century were more easily observed than those of our modern automobile, but the cycloid is still a curve that is readily generated and one in which many students of all ages easily take an interest. In a variety of settings, when I have mentioned, for example, the path of an ant riding on the side of a bicycle tire, some immediate interest has been sparked (see Figure 2.13a).

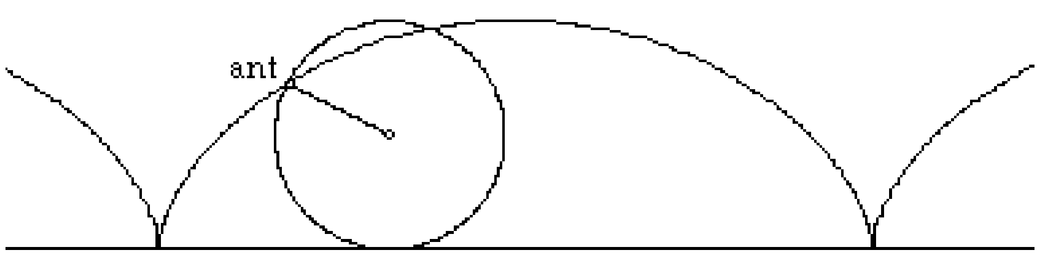

Figure 2.13a

The cycloid played an important role in the thinking of the seventeenth century. It was used in architecture and engineering (e.g. Wren's arches, and Huygens' clocks). As analytic methods were developed, their language was always tested against known curves, and the cycloid was the preeminent example for such testing (Whitman, 1946). Galileo, Descartes, Pascal, Fermat, Roberval, Newton, Leibniz and the Bernoullis, as well as the architect, Christopher Wren, all wrote on various aspects of the cycloid. Besides the fact that it can be easily drawn, what makes this curve an excellent example for this discussion is that its areas, tangents, and arc-lengths were all known, from the geometry of its generation, many years before Leibniz first wrote an equation for the curve in 1686 (Whitman, 1946).

Some early observers thought that perhaps the cycloid was another circle of a larger radius than the wheel which generated it. Some careful observation will dispel this belief; for example at the cusps where the traced point touches the ground the tangents are already vertical, but this section of the curve is clearly not a half circle.

Galileo used the curve as a design for the arches of bridges. For this reason he sought to determine the area under one arch of a cycloid. He approached the problem empirically by cutting the shape out of a uniform sheet of material and weighing it. He found that the shape weighed the same as three circular plates of the same material cut with the radius of the wheel used to draw the curve. Galileo tried this experiment repeatedly and with care, and found again that the ratio of the area of the cycloidal arch to that of the wheel which drew it was three to one. He suspected however that the ratio must be incommensurable, probably involving π , and abandoned further attempts to more accurately determine the ratio (3:1 is correct as we shall see). Galileo gave the name "cycloid" to the curve, although it has also been known as a "roulette" and a "trochoid" (Struik, 1969; Whitman, 1946).

A French mathematician, Gilles Personne de Roberval (1602 - 1675), wrote a tract in 1634 that included both the area and tangent properties of the cycloid (Struik, 1969). This work was done just before the publication of Descartes' *Geometry*, and several important issues are raised by Roberval's mechanical methods which involve no algebra. He began by imagining a point P on a wheel drawing a cycloid, and at the same time observing a second point Q drawing a second curve which he called the "companion of the cycloid." This second point Q has, at all times, the same elevation off the ground as P, but always rides on a vertical diameter of the wheel. Q can be thought of as the projection of P onto the vertical diameter of the wheel. See Figure 2.13b which shows both curves traced by *Geometer's Sketchpad*. Q will move forward at a constant speed while monitoring the height of P. The path of Q is, therefore, what is now know as a sine or cosine curve.

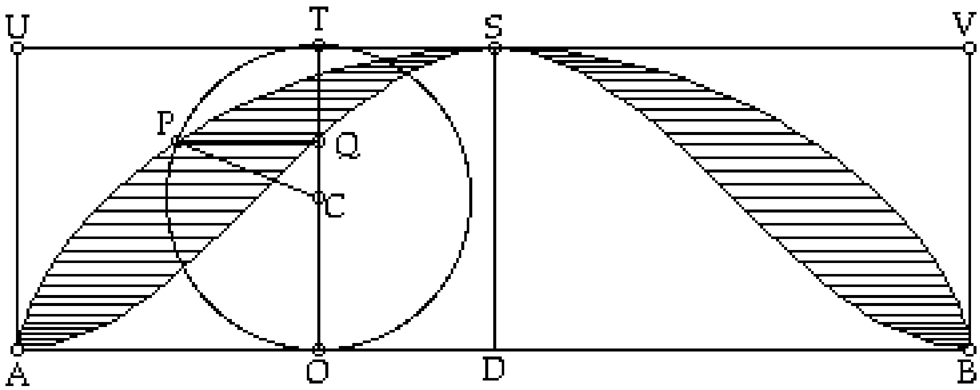

Figure 2.13b

Points P and Q start together at A, and come together again at S. In between the distance between P and Q takes on all the different horizontal segments that occur in half of the circle (i.e. all of the horizontal line segments $\overline{PQ}$ that form the shading). Thinking of the shaded area between the curves from A to S as a deck of cards, if one pushes them against a vertical line, they will form a half circle. Hence the entire shaded area in figure 2.13b is equal to the area of the circle. This reasoning employs what is known as the method of Cavalieri, also known as the method of indivisibles.

Looking at the symmetry of the companion curve traced by Q between A and S told Roberval that the area under that curve is one half the area of the entire rectangle $ABVU$. The entire rectangle has dimensions equal to the diameter and the circumference of the wheel, and is therefore equal to four times the area of the wheel (i.e. $(2\pi\, r)(2r) = 4(\pi\, r^2)$). The area under the cycloid is the shaded area plus the area under the companion curve, and therefore equals three times the area of the wheel that generated the curves, just as Galileo's weighing experiments had indicated.

Another way of stating this result is to say that the area of the cycloidal arch is always $3/4$ of the rectangle that contains it. Other mathematicians of the time (e.g. Wallis and Newton) would have called three quarters the characteristic ratio of the curve (Dennis & Confrey, 1993). This tradition goes back to ancient mathematics, like the result of Archimedes that says that if the curve under consideration was any downward parabola then the area under the curve would be $2/3$ of the rectangle containing it.

Roberval obtained tangents to the cycloid by thinking of the motion of point P as two separate motions, one rotational and the other forward (Struik, 1969). Since the wheel is rolling smoothly without slipping the rotational speed of the wheel must equal its forward speed (see Figure 2.13c). One can then construct the tangent as the sum of these two equal velocities. Thus Roberval constructed the tangent at P by considering a tangent to the circle at P (PH=rotational velocity), and a horizontal of the same length (PQ=forward velocity), and then forming the parallelogram on these two segments, and then drawing the diagonal $\overline{PV}$. Since $PH = PQ$, $\overline{PV}$ will bisect the angle $\angle HPQ$.[1]

[1] Roberval applied this same method of finding tangents by components to the parabola and the ellipse. For example a point on a parabola is increasing (or decreasing) its distance from the focus at the same rate as it is increasing (or decreasing) its distance from the directrix. Bisecting the angle, or drawing the diagonal between appropriate equal segments will yield the tangent. This is nearly the same tangent construction as van Schooten's (see Section 2.4).

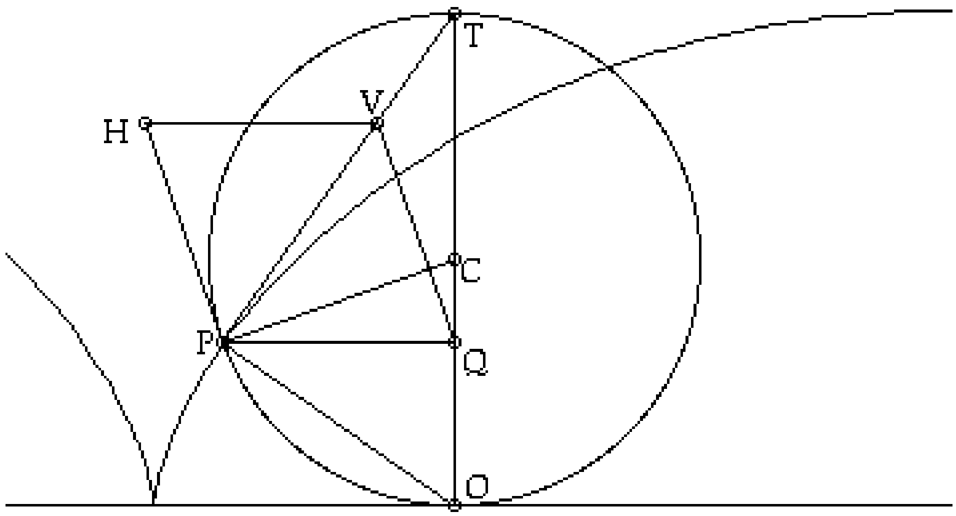

Figure 2.13c

Since $\overline{PH}$ is perpendicular to the radius $\overline{CP}$, and $\angle CPT = \angle CTP$ (isosceles triangle), and $\angle TQP = 90°$, then $\angle HPV = 90° - \angle CPT = \angle TPQ$. Hence the bisector $\overrightarrow{PV}$ of angle $\angle HPQ$ lies along the line $\overleftrightarrow{PT}$. One can deduce from this geometry that this tangent $\overleftrightarrow{PV}$ to the cycloid at P always points at the top of the rolling circle T. Look back at Figure 2.13b to see the tangent $\overleftrightarrow{PT}$ in another position. Thus the ant on the bicycle wheel is always moving directly towards or away from the top of the wheel.

One can also deduce from Figure 2.13c that the tangent to the cycloid is always perpendicular to the line $\overleftrightarrow{PO}$ which connects P to the point of contact of the wheel with the ground. In 1638, Descartes saw this directly by approaching the tangent problem in a different way. Instead of a circular wheel, he started by imagining a rolling convex polygon (e.g. a square wheel). Such a figure pivots on one vertex until a side comes down flat on the ground and then it shifts to pivot on the next vertex. Thus any point P, moving on a rolling polygon, will have as its path a series of circular arcs of different radii. While the polygon is pivoting on any one vertex, the path of that point P will be a circle centered at that vertex, and thus its tangent will be perpendicular to the line connecting P to that vertex (i.e. the point of contact with the ground). Descartes then imagined regular polygons with an increasing number of sides, becoming closer and closer to a circle. From this he deduced that the tangents at each point P on a cycloid must always be perpendicular to the segment $\overline{PO}$ which connects that point P with the point of contact O of the wheel with the ground (Whitman, 1946).

The approaches of Roberval and Descartes to this problem display their different conceptions of mathematics. Roberval thought in terms of engineering and mechanics. He saw the cycloid as two combined motions and resolved them using the parallelogram law, in the manner of Galileo. Descartes' approach is more geometrical, and involves seeing a circle as a limit of polygons (a ancient view taken, for example, by Archimedes). Descartes called the cycloid one of the "mechanical" curves that he refused to admit to his *Geometry*, because the regulation of its motion was not "clear and distinct" (i.e. it involved matched simultaneous rotation and forward motion).

If a wheel rolls at a constant rate, both of these approaches will yield not only the tangent to the path of motion at each point (i.e. the direction of velocity), but also the magnitude of the velocity vector as well. With Roberval's construction, if the wheel is rolling at a constant rate, then the horizontal velocity has constant magnitude, and by adding it to a vector tangent to the circle, and of the same magnitude as the horizontal velocity; one can, at all points, construct the cycloidal velocity vector. Using Descartes' conception of polygonal rolling motion, and thinking of the rotational rate at each pivotal contact point as constant, one can see that the magnitude of the velocity vector is proportional to the distance of the moving point from the contact point. This will remain true as the polygons approach the circle.

One sees, in either case, that the velocity is zero at the cusp of the cycloid when the point P touches the ground, and twice the forward velocity of the wheel when P is at the top of the wheel. Using Roberval's conception, this can be nicely animated using *Geometer's Sketchpad* (see Figure 2.13d). At the point of contact with the ground the two motions (rotational and forward) cancel each other, and the velocity vector is zero. At the top they are both in same direction and the velocity is at its maximum of double the constant forward speed of the wheel.

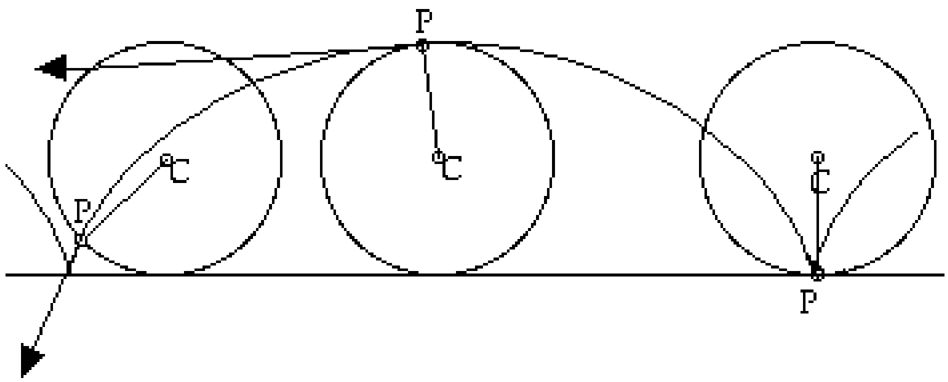

Figure 2.13d

As the seventeenth century progressed interest in the cycloid intensified and a variety of mathematical, physical, and engineering questions were investigated by Pascal, Huygens, Leibniz, Bernoulli and others (Arnol'd, 1990; Whitman, 1946; Whiteside, 1961; Smith, 1959). I will present one more investigation of the cycloid that gives the arclength of any portion of the curve in a simple geometric form. I first found this "rectification" (i.e. the finding of a straight segment equal to a given arclength) in the early notebooks of Newton from 1668 (1968, p. 193), but it also appears in a tract by John Wallis of 1659 (1972, p. 536). It was attributed by Newton to the famous London architect Sir Christopher Wren from a tract written in 1658 (Newton, 1968; Whitman, 1946). Like Galileo, Wren saw the cycloidal arch as well suited for architecture.

We have already seen that for any point on the rim of a rolling wheel, the segment that connects the point with the top of the wheel is tangent to its path of motion. Wren showed that the length of this segment is always exactly one half of the arclength between the point and the top of the cycloidal arch on which it is moving. That is to say in Figure 2.13e, the length of segment $\overline{QT}$ is exactly on half the arclength between P and T.

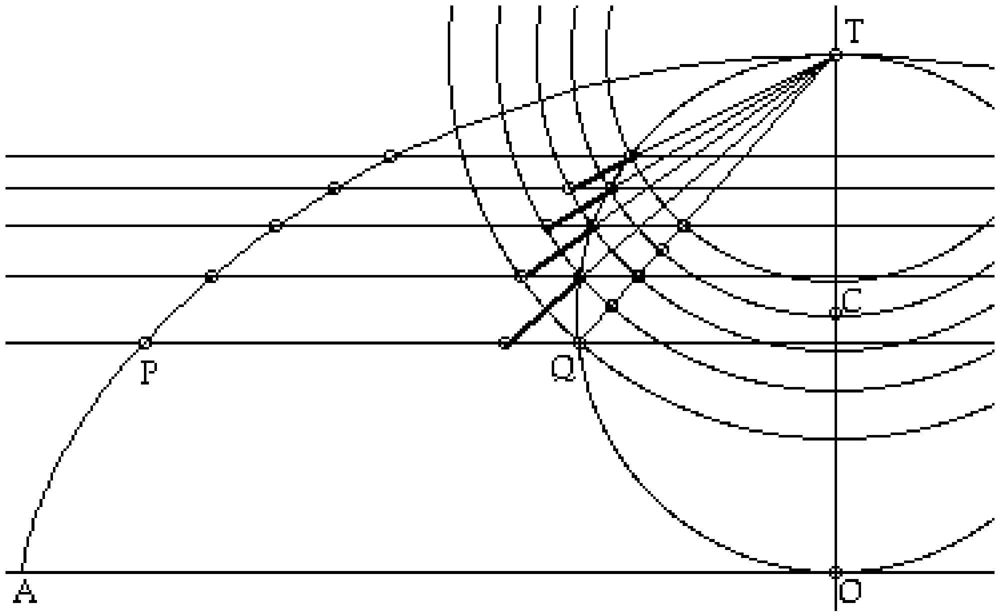

Figure 2.13e

$\overline{QT}$ is parallel to the tangent at P. Wren, like others in his time, imagined a curve to be made up of small line segments. Wren then imagined a series of points along the curve. Lines parallel to their tangents are shown radiating from T, and a series of circles, centered at T, pass through the intersection of these lines with the circle TQO. Each of the small darkened line segments is equal to a small tangent segment to the curve. Figure 2.13e then shows that the segment $\overline{QT}$ is the sum of pieces each of which are half of one of these tangential pieces of arclength. Thus twice QT must equal the entire arclength from P to T.

I find this theorem startling in its simplicity, especially after having calculated arclengths using the integral formulas from a calculus book. Wren presented his argument in the turgid formal Greek style known as the "method of exhaustion," but Newton provided only slightly more than what I have already said (1968, p. 193). The use of such methods was becoming quite natural to Newton (and also to Leibniz as we shall see in the next section).

This arclength property implies that the length of one entire cycloidal arch is exactly four times the diameter of the wheel which generated the curve. The circumference of the wheel is π ($\approx$ 3.14) times the diameter. For a point to traverse one cycloidal arch the wheel must revolve once. The extra distance that is added by the forward motion stretches the path of motion from <u>π diameters</u> to <u>4 diameters</u>. It is interesting to think back to the ant on the rim of a wheel. On the upswing, her motion is always headed straight for the top of the wheel, but the length of her cycloidal path to the top will always be twice her distance from the top at any given moment.

I would ask the reader to reflect here on the things which can be known about curves solely from considering the actions which produce them. An equation for the cycloid was not written down until after all of the above discussions. When I think of how, in the past, I have presented this curve in my calculus classes using the standard parametric equations, I feel that both I and my students learned very little. In the secondary curriculum, cycloids are rarely mentioned, because their equations are considered too difficult.

What is governing our choice of curriculum? It would seem to be regulated by algebraic convenience. Students are asked to consider many curves that I have never seen in daily life, simply because their equations are tractable. Analytic methods are powerful tools, but letting the tools govern the subjects of our thoughts can only lead to tedious and unnatural formalism. As Leibniz labored to create the language and notation that we call calculus, he had to test this language to see that it was consistent with what was known about areas, tangents, and arclengths. Curves such as the cycloid were used as critical experiments to test the validity of linguistic constructions. Leibniz first wrote an equation for the cycloid in 1686, and then used it to test his evolving notations (Whitman, 1946).

Leibniz wanted to create a universal language which was capable of expressing all known results about areas, tangents, arclengths, and other quantities. Newton accused Leibniz of plagiarism, because he never came up with any previously unknown answers to questions about areas, volumes, tangents, or arclengths. Newton misunderstood the intention of Leibniz. He was not a plagiarist; he was a linguist. He largely succeeded in his quest for a universal language capable of expressing all of the known results from the geometry of his day.

It is this sense of language as a human construction, evolving from experiences, and fitted to certain purposes, that I want suggest should be brought into the mathematics classroom. By eliminating the discussion of curves like the cycloid, the grounded activity which justifies language construction is taken away from students. They then have no critical experiments upon which to test the consistency and validity of the formalisms they learn. They learn only about the nature of representations that refer to themselves in an endless hall of mirrors. Students often see mathematics as perfect and unquestionable, because they have only experienced it within a self-referential frame. Right at the very beginning of the scientific revolution, Pascal objected to this general linguistic trend in modern thought. Speaking theologically he said, "Nature possesses forms of perfection in order to show that it is an image of God; and faults to show that it is only an image" (Pascal, 1962, #262). I will take the liberty to paraphrase him and say that: Mathematics possesses forms of perfection in order to show that it is an image of Nature; and faults to show that it is only an image.

References for this article can be found at Mathematical Intentions.