Newton and Empirical Interpolation

In the second of the two letters to Oldenburg of 1676, Newton remarks, "When simple series are not obtainable with sufficient ease I have another method, not yet published, by which the problem is easily dealt with. It is based upon a convenient, ready, and general solution of the problem. To describe a geometrical curve which shall pass through any given points. . . Although the problem may seem to be intractable at first sight it is never the less quite the contrary. Perhaps indeed it is one of the prettiest problems that I can ever hope to solve." (Fraser, 1927, p.45). By the term "geometrical curve" here Newton means a curve with a polynomial equation. He did eventually publish (1710) his method for finding a polynomial equation that would pass through any given finite set of points in his *Methodus Differentialis* (Newton, 1967b, Vol. 2).

We shall briefly describe one section of this work and show its connection to the binomial series. We seek to show how important finite difference tables were in Newton's work, and to emphasize the empirical groundwork that formed the basis of much of his thinking. As the quote above makes clear, finding say a fourth degree polynomial that passed through a given set of five data points was for Newton the same problem as generating the first five terms of an infinite series. Theoretical expansions and fitting a curve to data were in his mind the same problem. For more details see the excellent article by Fraser (1927).

Newton first presented his method by working out in detail the example of finding a fourth degree polynomial equation that passes through an arbitrary set of five points. He first calculated what he called "divided differences." We shall employ Fraser's notation and make a table to show how these differences are calculated. Let Δ' denote the first divided difference which is the same as the average rate of change between two points. Δ'_2 , Δ'_3, Δ'_4 , shall denote the second, third, and fourth divided differences. Each of these is calculated as the difference in the previous one divided by the overall difference in the x-values which were involved in its formation. Three data points are needed to construct one value of Δ'_2, four are needed for each value of Δ'_3, etc. See Table 11. Note that Newton calculated his differences in the opposite order from most modern conventions, but since he is dividing by the x differences his signs will come out the same.

Table 11

x	y	Δ'	Δ'_2	Δ'_3	Δ'_4
p	α				
		$\frac{\alpha-\beta}{p-q}$			
q	β		$\frac{\Delta'(p,q)-\Delta'(q,r)}{p-r}$		
		$\frac{\beta-\gamma}{q-r}$		$\frac{\Delta'_2(p,q,r)-\Delta'_2(q,r,s)}{p-s}$	
r	γ		$\frac{\Delta'(q,r)-\Delta'(r,s)}{q-s}$		$\frac{\Delta'_3(p,q,r,s)-\Delta'_3(q,r,s,t)}{p-t}$
		$\frac{\gamma-\delta}{r-s}$		$\frac{\Delta'_2(q,r,s)-\Delta'_2(r,s,t)}{q-t}$	
s	δ		$\frac{\Delta'(r,s)-\Delta'(s,t)}{r-t}$		
		$\frac{\delta-\varepsilon}{s-t}$			
t	ε				

Given five points only one value of Δ'_4 can be calculated. Hence we seek a function which has that constant value as its fourth divided difference everywhere. Such a function must be a fourth degree polynomial. We then must find its coefficients. Here is how Newton described that procedure. Suppose that $y=a+bx+cx^2+dx^3+ex^4$. When we form first differences, 'a' drops out. When we divide those differences by the changes in x, the denominator must be a factor of the numerator and therefore it can be canceled. This happens at each step, i.e. we lose one more of the polynomial coefficients and the divisors are always factors. Newton wrote the divided differences in his table in polynomial form as follows:

(18)

$$\Delta'(p,q) = b+c(p+q)+d(p^2+pq+q^2)+e(p^3+p^2q+pq^2+q^3)$$
$$\Delta'(q,r) = b+c(q+r)+d(q^2+qr+r^2)+e(q^3+q^2r+qr^2+r^3)$$
$$\Delta'(r,s) = b+c(r+s)+d(r^2+rs+s^2)+e(r^3+r^2s+rs^2+r^3)$$
$$\Delta'(s,t) = b+c(s+t)+d(s^2+st+t^2)+e(s^3+s^2t+st^2+t^3)$$

$$\Delta'_2(p,q,r) = c+d(p+q+r)+e(p^2+pq+q^2+pr+qr+r^2)$$
$$\Delta'_2(q,r,s) = c+d(q+r+s)+e(q^2+qr+r^2+qs+rs+s^2)$$
$$\Delta'_2(r,s,t) = c+d(r+s+t)+e(r^2+rs+s^2+rt+st+t^2)$$

$$\Delta'_3(p,q,r,s) = d+e(p+q+r+s)$$
$$\Delta'_3(q,r,s,t) = d+e(q+r+s+t)$$

$$\Delta'_4(p,q,r,s,t) = e$$

Here we see that the fourth divided difference is the coefficient of the fourth degree term. Knowing *e* we can now go back to either of Δ'_3 equations and solve for *d*. Knowing *e* and *d* we can go back to any one of the Δ'_2 equations and solve for *c*. Continuing in this way we can completely determine the desired polynomial. As in his previous table interpolations, Newton proceeded by solving a set of linear equations. Note that once the table of divided differences is made, only one value from each column is needed in order to construct the polynomial. Newton discussed a variety of strategies for using either the lead differences or a set of central differences, depending on the nature of the data involved (Newton, 1967b; Fraser, 1927).

We shall make only one more observation here. Consider the special case were the values of x are evenly spaced at unit intervals beginning with 0. In this case all of the denominators in the Δ' column are 1, and all of the denominators in the Δ'_2 column are 2, and so on. In this case the each divided difference equals the ordinary finite difference divided by n factorial, i.e., $\Delta'_n = \frac{\Delta_n}{n!}$. In this case his fitted polynomial equation will come out as:

$$(19) \qquad y = \alpha + x\,\Delta 1 + \frac{x(x-1)\Delta 2}{2!} + \frac{x(x-1)(x-2)\Delta 3}{3!} + \frac{x(x-1)(x-2)(x-3)\Delta 4}{4!} \quad .....$$

$$y = \alpha + x\Delta_1 + \frac{x(x-1)\Delta_2}{2!} + \frac{x(x-1)(x-2)\Delta_3}{3!} + \frac{x(x-1)(x-2)(x-3)\Delta_4}{4!} + \ldots$$

Newton did not write his polynomials in exactly this algebraic form (19), but instead described in detail procedures for how to work from a table of differences. His procedures do imply this form (Fraser, 1927). Looking at (19) one can see both the general form of the binomial coefficients and the form of the Taylor series. Taylor took his inspiration from Newton and wrote his derivations based upon difference tables in 1715 (Callinger, 1982, p.419).

References cited in the text can be found at
http://www.quadrivium.info/MathInt/Notes/WallisNewtonRefs.pdf