Similarity, Geometric Arithmetic, and the Geometric Mean

MATHEMATICAL INTENTIONS

This section addresses some topics that are in the high school geometry curriculum, but are not always covered explicitly.

Before the institutionalization of the calculus by Euler, problems of finding tangents to curves were usually solved using similarity and geometric means.

One of the main subjects of these lectures is the algebraization of geometry. Section D fills in the reverse process: doing arithmetic with geometric constructions.

A. Similar triangles

Intuitively, similar figures are those that have the same shape, but (possibly) a different size. A more precise definition includes the idea of a *dilation*, also called a similarity transformation. The idea is that you choose a "center" point, and every point of the figure is stretched (or shrunk) outwards from that point.

Definition. To *dilate* a point A from a center D, by a factor of r,

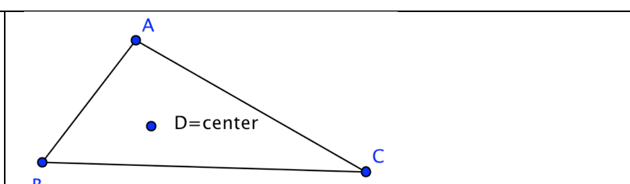
i) Draw a ray from D through A.

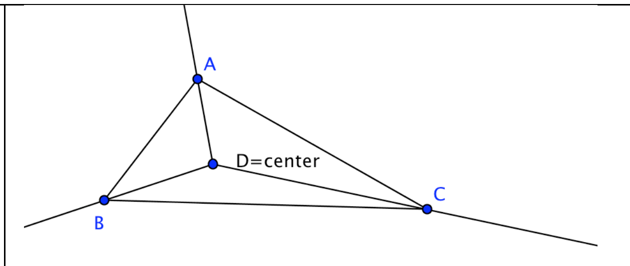
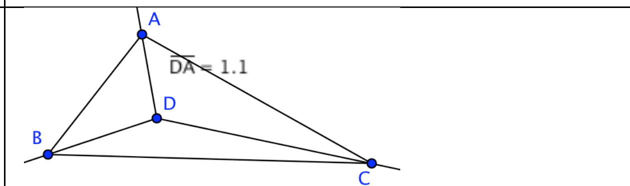
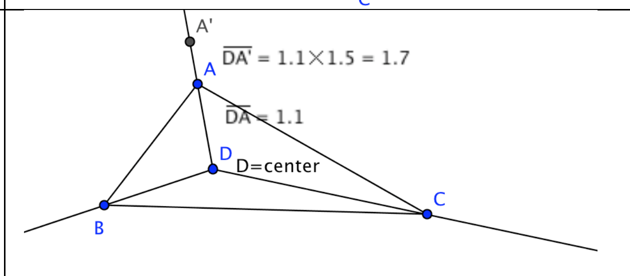
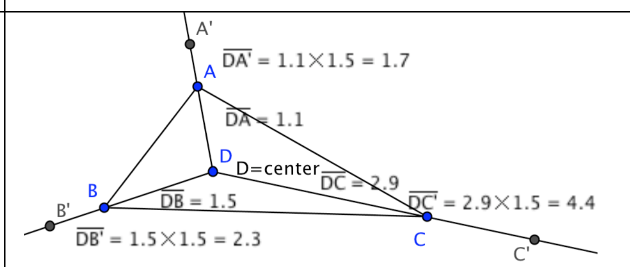
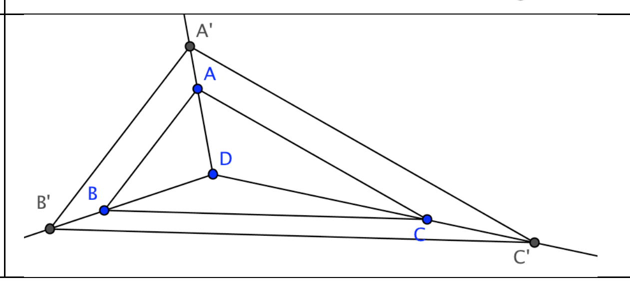
ii) Measure the distance from D to A, and multiply it by r.

iii) Measure out this distance from D and mark point A′. A′ is the dilation of A from D by a factor of r.

To dilate a set of points, dilate all the point in the set. You can prove that a dilation of a line is a line, and a dilation of a polygon is a polygon.

Example 1. Here are the steps to scale a triangle by a factor of 1.5. Keep in mind that everything starts at D: the rays, the original measurements, and the multiplied measurements.

Mark a "center" point, D, for the dilation. It can even be outside the figure or on the figure. (See later examples.)	

Draw rays from the center through the vertices of the figure. For scale factors greater than 1, dilating will move the vertices outwards along these rays. For scale factors less than 1, the vertices will move inwards towards D.	
Measure the distance from D to A. In this example, the distance is 1.1.	
Multiply distance DA by the scale factor (1.5 in this example.) $DA \times \text{scale factor} =$ $1.1 \times 1.5 = 1.7$ (rounded to nearest 0.1) Measure this distance out from D for the position of A′.	
Repeat for B: measure from D to B, multiply distance by the scale factor 1.5, then measure out from D by the new distance. Repeat for C.	
Connect A′, B′, and C′ to get the scaled triangle.	

Try this (1). Dilate a triangle by hand.

a) Cut out a triangle, get some big sheets of paper and a ruler, and scale the triangle by various factors.

b) Superimpose the cutout on the finished dilated triangle to check that the angles are equal.

c) Dilate the triangle with a fixed scale factor (such as 2), but with various centers: inside the triangle, outside the triangle, a point on the triangle. Cut out one of the dilated triangles and superimpose it on the others to check directly that the point chosen doesn't affect the final shape or size.

Try this (2). Compare with other methods.

a) In Euclidean geometry, the Side-Side-Side congruence theorem guarantees that if two triangles have all pairs of corresponding sides equal, they are congruent. Suppose you measure the sides of your triangle, multiply them by the scale factor, then use a ruler to (try to) construct a triangle with these side lengths. How does the accuracy of this method compare with the results using a dilation? Why?

b) In Euclidean geometry, the Side-Angle-Side and Angle-Side-Angle congruence theorems guarantee that if two triangles have all pairs of corresponding parts equal (two sides and an included angle, or vice versa), they are congruent. Suppose you measure the sides and angles of your triangle, multiply the side(s) by the scale factor, then use a ruler and protractor to construct a triangle with these side lengths. How does the accuracy of these methods compare with the results using a dilation? Why?

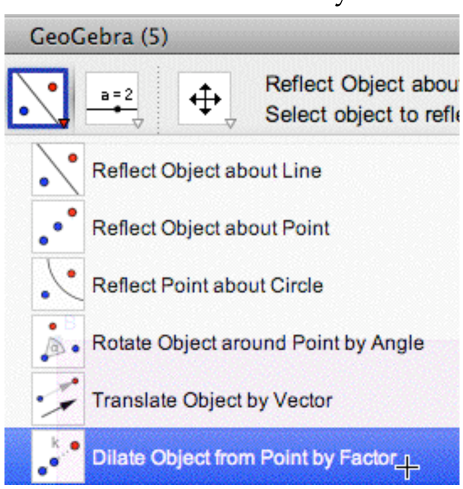
Try this (3). Experiment with the GeoGebra files:

a) SimTri.html has adjustable center, triangle, and scale factor, and calculates distances and ratios in the spreadsheet window.
   i) What does the configuration look like if the center is inside the triangle?
   ii) What does the configuration look like if the center is one of the vertices of the triangle?
   iii) What happens with a negative scale factor? Explain why.

b) SimQuad.html dilates an adjustable quadrilateral

c) SimPenta.html dilates an adjustable pentagon.

d) Download a copy of the program (or just double-click in the geometry window of one of the files above) and dilate some polygons and circles using the dilation tool. The dilation tool is the on the same button as the reflection tool (third from the right), and at the bottom of the list when you hold down the small triangle.



Definition. Two figures are similar if they can be positioned so that one is a dilation of the other. The number r described above is called the scale factor.

Note that these definitions work in three (or more) dimensions.

Theorem on similar triangles from Euclidean geometry.

(a) If two triangles are similar, then corresponding angles are equal, and corresponding sides are proportional (with factor r.) In other words, dilations change lengths (each length is multiplied by r), but do not change angles.

(b) The Angle-Angle similarity test. If two triangles have two pairs of corresponding equal angles, then the triangles are similar.

Try this (4). For polygons with more than three sides, is part (a) of the theorem true? Is part (b) true?

B. Right triangles and similar subtriangles

Start with a right triangle. Drop a perpendicular to the hypotenuse from the vertex with the right angle. This divides the triangle into two smaller triangles. Each is similar to the original. (Why?)

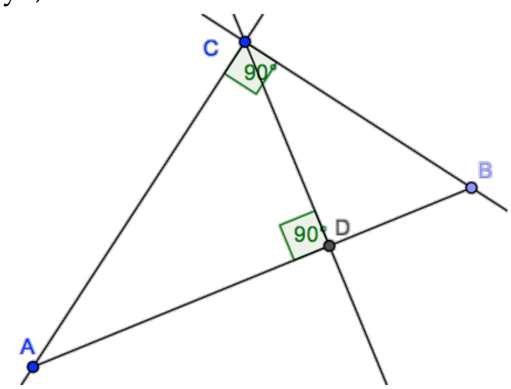


Figure 1. Similar right triangles.

Does this work for triangles that are not right triangles? For an arbitrary triangle, can you find a line that cuts off a similar triangle?

C. Triangle inscribed in a semicircle

Theorem.

(a) Choose the endpoints A and B of a diameter of a circle and one other point, C, on the circle. Then ABC is a right triangle.

(b) Conversely, start with a right triangle. Find the midpoint of the hypotenuse. Construct a circle for which the hypotenuse is a diameter. The circle will go through the third point: the vertex at the right angle.

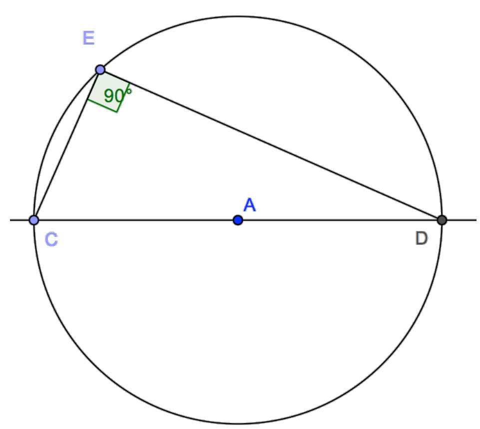


Figure 2. A right triangle inscribed in a circle.

Prove it yourself or look it up in a high school geometry book. Hint for part (a): use similar triangles as in section B, above, and the fact that the sum of the angles in a triangle is 180 degrees.

Try this (5). Open the file SimRtTri.html. Experiment with triangles inscribed in a semicircle (the three vertices are on the circle, and one side is a diameter of the circle) and general inscribed triangles (the three vertices are on the circle). What can you say about the angles?

D. Arithmetic with straightedge and compass

See the lecture notes on Constructions, if you have not already read it.

Arithmetic is about numbers and operations. Geometry is about space and shapes. One of the connections between the two is measurement: given a geometric object, measuring some quantity associated with the object, with a chosen unit of measure, gives a number.

These lectures are about the transition from mathematics as being principally about geometry to being principally about numbers and their syntax. Part of this transition is proving that you can do arithmetic with geometry. (In the 17th century, mathematicians proved you can do essentially all geometry with arithmetic.)

Try this (6). Arithmetic with straightedge and compass.

You can do any calculation on a 5-function calculator with straightedge and compass. Because a straightedge is not marked with numbers, you need to *prove* that you can do arithmetic with these tools. Specifically, suppose you are given any two line segments A and B, and a unit segment, defined to have length 1. The unit allows you to measure the segments to obtain two positive real numbers $a = \text{length(A)}$ and $b = \text{length(B)}$.

Give a general method for constructing segments of these lengths:

a) $a+b$

b) $a-b$

c) ab

d) $\frac{a}{b}$

e) $\sqrt{a}$

Given segments A and B, it's easy to make a rectangle of area ab. That's not what's happening here; you are to construct a *segment* of *length* ab. Similarly, in this exercise, division of two lengths should give a length, and the square root of a length should be a length. Ordinarily, the square root of an area of a square is the length of the side of the square.

In this way, you can make a marked number line that includes integers, rational numbers, combinations of rational numbers with square roots of integers, square roots of these, and so on.

Hints for c) and d): use similar triangles that include sides of lengths a, b, and 1.

Hint for e): Use sections B and C above. Use the Pythagorean theorem on the triangles ABC, ACD, and BCD in section B to find the length of CD in terms of AD and BD. Then use AD=a and BD=1 and construct ABC.

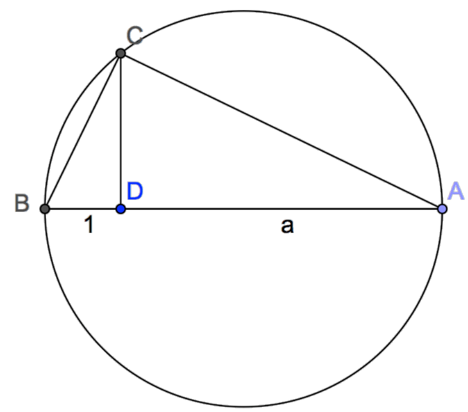


Figure 3. A straightedge and compass construction of $\sqrt{a}$. (Where is $\sqrt{a}$?)

Try this (7). Once you have worked on the square root construction (v), experiment with the GeoGebra file SqrtConstr.html.

E. The geometric mean

In the period studied in these lectures, ratios and proportions were often considered geometrically, as ratios of lengths. There were methods for constructing lengths with given ratios to existing lengths.

One common type of problem was "Find the mean proportional between a and b."

This means: find a length c so that the ratio of a to c is equal to the ratio of c to b:

$$a:c = c:b \qquad \text{or} \qquad \frac{a}{c} = \frac{c}{b}$$

Modern algebraically trained people prefer second form, and transform it by cross-multiplication to

$$ab = c^2$$
$$c = \sqrt{ab}$$

Definition. The *geometric mean* of two numbers is the square root of their product. This is the multiplicative analog of the (additive) *arithmetic mean*, or average: half the sum of the numbers. The geometric mean figures prominently in the construction of logarithm tables.

The geometric mean of two lengths, a and b, can be constructed with straightedge and compass. Just generalize the diagram above, replacing the unit length with the length b.

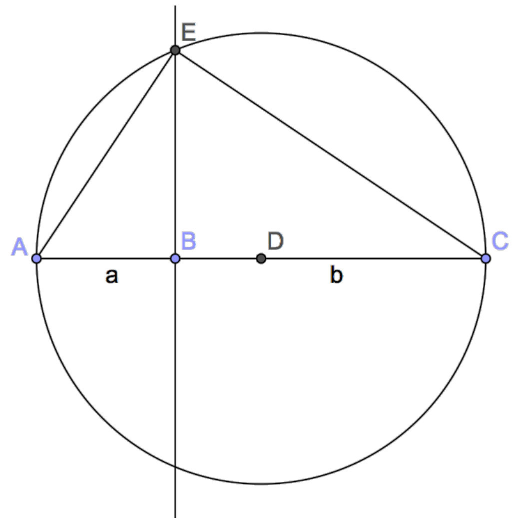


Figure 4. Construction of the geometric mean.
(Where is the geometric mean of *a* and *b*?)

Try this (8). Use straightedge and compass or a dynamic geometry program to construct the geometric mean of two lengths. Measure with a ruler (or the geometry program), and check with a calculator that BD really is the geometric mean of AB and BC.

Try this (9). Note that it's not necessary to know what the unit length (1) is to do this construction, whereas multiplication, division, and square root *did* need a segment of length 1. Addition and subtraction do not need a unit. Why?

F. List of applets used in this section

SimTri.html
SimQuad.html
SimPenta.html
SimRtTri.html
SqrtConstr.html