2.2. Fractions, part 1

You may be wondering why we're doing fractions before spending much time on whole numbers. The reason for this is that fractions are harder to learn and teach than whole numbers. You, as an adult, are very experienced with whole numbers, and don't need as much review to teach them successfully. Even if you are an expert with fractions, you will probably learn some new (and some very old) things in this section. Remember, the order of topics in this book is not necessarily the way they would be sequenced in a curriculum for children, who are learning them for the first time.

This section introduces fraction concepts using the length model: showing the size of a fraction by length. Section 2.3 introduces other models for fractions, such as area, volume, and angle, including the familiar "pie" model.

A. Fractional units and unit fractions

We often think of units of measure as small quantities that measure larger quantities. This way there is a whole number of units that make up the larger quantity you are measuring, and you can just count the units. For instance, there are 12 fluid ounces of volume in a standard soft drink can. The length of a sheet letter-size (U.S.) paper is 11 inches. An hour is 60 minutes.

But units go the other way, too: small quantities are sometimes measured in units that are larger than themselves. For example, for a half gallon of milk, the unit of measure of volume is a gallon, which is bigger than the quantity you are measuring. A running track is 1/4 mile. Here the quantity being measured is the length of the track, and the unit is a mile. If you say you'll meet someone in a half an hour, the unit is an hour, and the amount of time being measured is smaller than the unit.

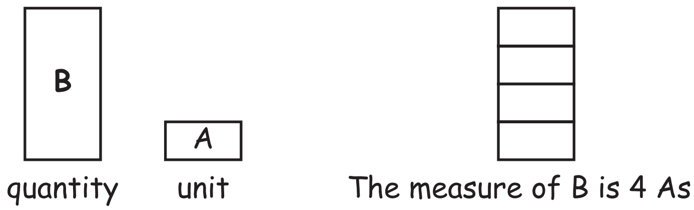

Figure 1. Quantity B is measured with a unit, A, smaller than itself. The measure of B is 4 As.

Suppose the quantity B is being measured with the unit A, and a whole number (n) of copies of A fill B.

Now switch the roles of the quantity and the unit: A is the quantity and B is the unit

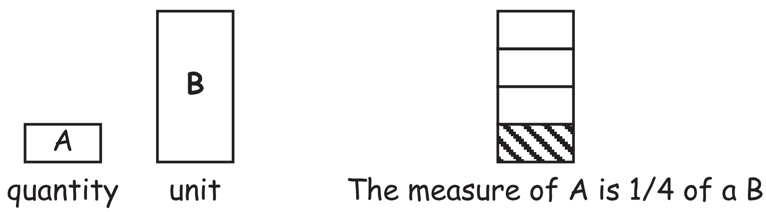

Figure 2. Quantity A is measured with a unit, B, bigger than itself. The measure of A is 1/4 of a B.

Definition: The meaning of the symbol $\frac{1}{n}$, where n is a positive whole number. If a smaller quantity A is being measured with a bigger unit B, and *n* A's fill B, then the fraction $\frac{1}{n}$ is the number of B's that fill A.

When n is a positive whole number, $\frac{1}{n}$ is a small number, between 0 and 1. The number $\frac{1}{n}$ is called a *unit fraction*.

The number n is called the *denominator* of the fraction. It gives the size, or denomination[1], of the fractional unit.

Example 1. Everyday quantities measured in unit larger than themselves.

a) 12 inches make a foot (a foot measured in units of inches), so an inch is $\frac{1}{12}$ of a foot (an inch measured in units of feet.)

[1] An online dictionary, WordNet, defines "denomination" as "a class of one kind of unit in a system of numbers or measures or weights or money: 'he flashed a fistful of bills of large denominations'"

b) A dollar is 100 pennies (a dollar measured in units of pennies), so a penny is $\frac{1}{100}$ of a dollar, or $.01 (a penny measured in units of dollars.)[2]

c) A liter is 1000 milliliters (a liter measured in units of milliliters), so a milliliter is .001 (that is, $\frac{1}{1000}$) of a liter (a milliliter measured in units of liters.)

Writing fractions with the fraction bar. Fractions can be written with a horizontal bar[3]: $\frac{1}{n}$, or with a diagonal bar[4] (slash): 1/3. The horizontal bar is more clearly understandable. For example, the hand-written symbol 2 1/3 might mean $2\frac{1}{3}$ or $\frac{21}{3}$. On the other hand, if you're typing into a computer program, there is usually no simple way to type a top number and a bottom number, but it *is* simple to type a fraction with a slash. Most computer programs, such as a spreadsheet or an online calculator, understand the slash as either a fraction bar or division (see section 2.6 for why a fraction bar and division are the same.)

Words for fractions. This table of names is included to emphasize the somewhat irregular way number words are formed in English. Children may not pick this up without instruction, especially those learning English as a second language. Also note that "third", "sixth", etc., could mean either a unit fraction or an *ordinal number* which gives a position in a sequence, as in "She came in third in the whole race!"

Denominator of unit fraction	**Unit fraction**	**Name, in words**
1	1/1	whole
2	1/2	half (plural halves)
3	1/3	third
4	1/4	fourth
5	1/5	fifth
N = 6 and up	1/N	[the number word] with "th", unless the number is more than 20 and the ones digit is 1, 2, 3, or 5: see examples of exceptions below.

[2] The word "cent" is the root for lots of words meaning 100 or 1/100.
[3] The horizontal fraction bar is also called a vinculum.
[4] The diagonal fraction bar is also called a virgule. Nobody uses these words.

21	1/21	twenty-first, not twenty-oneth
32	1/32	thirty-second, not thirty-twoth
53	1/53	fifty-third, not fifty-threeth
75	1/75	seventy-fifth, not seventy-fiveth

Fractions are used when a measurement doesn't "come out even", that is, is not a whole number of the main unit you are using. But usually one unit fraction isn't the exact amount extra, either. How do you combine unit fractions to measure a quantity with whatever accuracy you need? There are a number of strategies, some used in the past, some used now. In the next sections, we'll discuss three systems of fractions:

1. Binary fractions, based on halves, halves of halves, etc.
2. Common fractions: the kind with numerators and denominators
3. Decimal fractions, based on tenths, tenths of tenths, etc.

Information and activities about Egyptian fractions, another strategy for measuring in parts of a unit, are included in the problems. The ancient Egyptians used unit fractions in combinations, but at most one of each size of unit fraction.

B. A system of fractions: binary fractions

Halving, which is dividing something in two equal parts, is a concept that is common to people in all cultures, and to young children. After all, if you are splitting some food with another person, each person wants to make sure they are getting their fair share.

Definition: A *binary fraction* is a fraction obtained by cutting the main unit in half (two equal pieces), sometimes repeatedly, and combining pieces of these sizes.

The prefix "bi-" (or sometimes "bis-") usually means two, as in bicycle and biscotti[5].

In the Yup'ik Eskimo language, there are no words for fractions, except "half". In Yup'ik, you can also say "half of a half", "half of a half of a half", etc.

Class activity 1. A strip of binary fractions

Materials:

Strips of letter-size paper, cut across the paper, $8\frac{1}{2}$ inches long (the width of the paper), and 1 inch wide. Use a paper cutter to cut carefully. 1 strip per person, plus extras in case of mistakes.

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5 Biscotti are Italian cookies that are twice (bis) cooked (cotto): they are baked once as a long loaf, then they are sliced thinly and baked again.

Regular rulers marked in inches

a) Make a strip marked in binary fractions by accurately folding a strip of paper as described below. The unit of length in this part is the length of the strip.

   i) Fold it in half as shown, then open it up. What fraction of the length of the strip is each of the sections formed by the fold?

   ii) Fold each end to the previous fold mark, then open. Before you unfold it, predict how long one of the sections will be (what fraction of the length of the strip.) Then check by looking at the whole strip, counting, etc.

   iii) -v) Repeat folding steps until you have reached step v. At each stage, the strip should be divided into equal length pieces. The 5th stage of folding will make sections that are half of half of half of half of half of the strip. Predict what fraction you will have at each stage, then check.

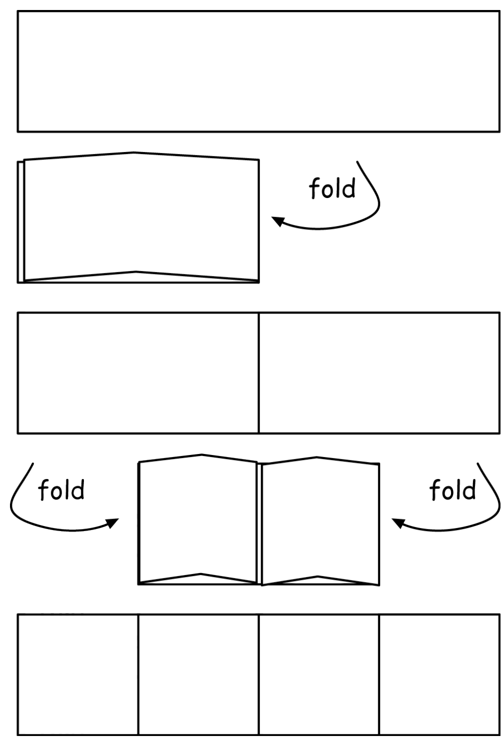
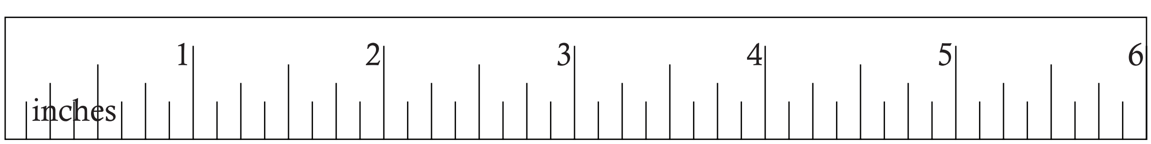


Figure 3. The first two folding steps, forming halves, then fourths.

b) Look at a regular inch ruler. The whole numbers of inches are marked with numbers and long marks. The fractional parts are not numbered, but have marks of shorter lengths. Identify what fractional units of an inch are between the various sizes of marks.



Rulers vary; some have more or fewer marks; some may not even be marked in binary fractions: check yours. Then practice until you can recognize the denominators of fractions of inches by the appearance and position of the marks.

C. Length manipulatives for fractions: fraction strips

Plastic fraction strips are a manipulative that can be purchased from math education supply companies. A set consists of strips, all the same width, and whose lengths are unit fractions. A set contains 1 whole, 2 halves, 3 thirds, 4 fourths, and so on. However, the commercial sets often skip some of the less popular denominators, such as sevenths. By making your own, you can have a complete set up to some denominator. Activities to make your own fraction strips by folding are included in the appendix to this section; these activities are highly recommended for young students to develop an understanding of fraction concepts. After doing these activities, you may use preprinted and precut strips, which your instructor may have provided.

You will have two copies of each preprinted strip: one to cut into pieces to line up with other pieces, and the other to mark as a ruler. Here is what your sixths will look like:

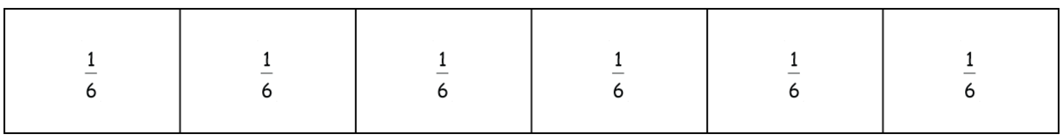


Figure 4. The unit strip, labeled in sixths, before cutting.

Cut into pieces:

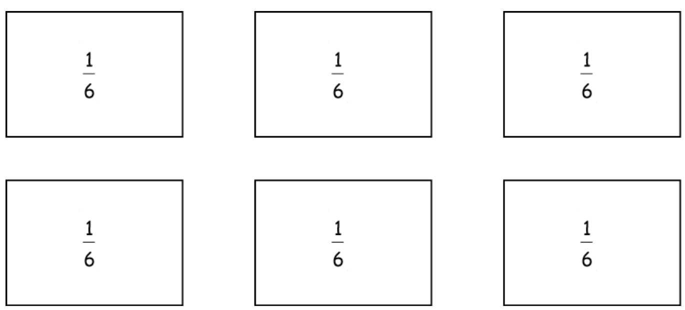


Figure 5. The 6 sixths, cut apart.

This is the sixths ruler:

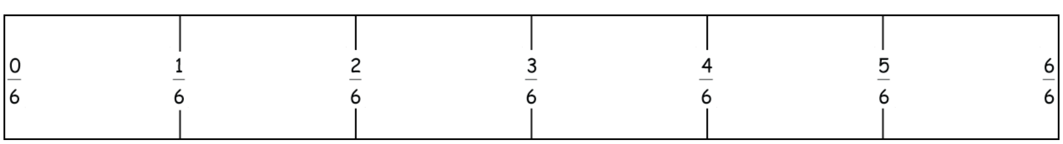
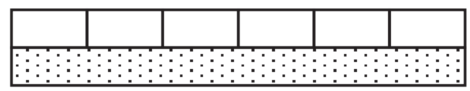


Figure 6. A second uncut strip, marked as a ruler in sixths.

In the activity, the set is constructed to have the width of a sheet of U.S. letter size paper as the unit of length. We'll refer to this length as a paperwidth. They will fit nicely into a business size envelope.

D. Another system of fractions: common fractions.

Common fractions are the kind with a numerator and denominator, like $\frac{3}{4}$, $\frac{8}{12}$, $\frac{7}{3}$, and $5\frac{3}{8}$. Common fractions use *only one* size of piece/unit fraction (such as sixths), but you are allowed to use many copies of that piece. So the name of a common fraction has two parts: one describes the size of the unit fraction, the denominator, and the other counts the number of that piece, the *numerator*[6].

Common fraction, 5/6: 5 copies of one denomination of piece, a sixth	
For reference: the unit 6/6 = 1	

In a common fraction, the numerator is written above the fraction bar (or to the left of the slash), and the denominator is written below the bar (or to the right of the slash.) A fraction out loud makes sense as a phrase: $\frac{3}{4}$ is read "three fourths", like three students, three cars, or three dollars.

The one exception to the rule that you must use only one kind of piece is for fractions greater than or equal to 1. In this case, you may simplify by trading smaller pieces for wholes. For example, $\frac{13}{4}$ is 13 pieces, four of which make a whole. You can make 3 wholes out of these pieces, with 1 left over: $3\frac{1}{4}$. This form of the fraction is read "three and three fourths"; the "and" means "+".

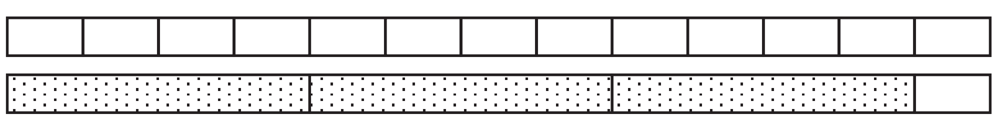


Figure 7. Top: 13 fourths. Bottom: 3 and 1 fourth.

It is certainly more convenient to deal with 3 larger and 1 small piece than 13 small pieces.

Definitions:

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[6] The word "enumerate" means to count.

A *proper*[7] *fraction* is a common fraction whose size is less than 1. This means that its numerator is less than the denominator.

Examples: $\frac{2}{3}$, $\frac{7}{10}$, $\frac{22}{33}$

An *improper fraction* is a common fraction whose size is 1 or greater, but is expressed (measured) with pieces of only one size of unit fraction, and no whole units.

Examples: $\frac{5}{4}$ (expressed only in fourths), $\frac{18}{12}$ (expressed only in twelfths), $\frac{14}{7}$ (expressed only in sevenths).

A *mixed fraction* is a common fraction that is 1 or greater, and expressed with a mixture of whole units and one size of unit fraction.

Examples: $1\frac{2}{3}$, $24\frac{7}{10}$, $4\frac{22}{33}$

Class activity 2. Measuring with common fractions.

a) Measure all the segments at the end of the section with fraction strips, using the rules for common fractions: only one size of piece. Choose the denominator that gives the most accuracy. Not all lengths can be measured exactly with your fraction strips; give an approximation if necessary. For example: between 5/7 and 6/7.

b) Also measure them with binary fractions: only pieces obtained by halving (wholes, halves, fourths, etc.). You can combine sizes.

Simplifying fractions

Definition: A fraction of any kind is *simplified,* or *reduced,* or in *lowest terms*, if it uses the fewest number of pieces, and the largest pieces that allowed in the system of fractions you are using.

For example, if you are using binary fractions, then $\frac{1}{4}+\frac{1}{4}$ is not simplified because two $\frac{1}{4}$ s have the same length as one $\frac{1}{2}$. That is, $\frac{1}{2}$ uses *both* bigger *and* fewer pieces than $\frac{2}{4}$.

Simplification of common fractions also means to use the fewest and largest pieces, keeping in mind the rules of the system. With common fractions, you may use only one kind of unit fraction: that is, only one denominator. So, when working with common fractions,

[7] It's unfortunate that the word "proper" sounds good and "improper" sounds bad. In fact, in some circumstances, improper fractions are more useful.

often you need to find a *common denominator*[8]*:* a single denomination/size/fractional unit that will allow you to combine all the parts of the number. The only exception is that you have the option of using improper fractions (only a unit fraction as unit) or mixed fractions (one size of unit fraction and wholes.)

Class activity 3. Simplifying and converting common fractions with fraction strips. Trade several small pieces for bigger piece, or vice versa (in part b.)

a) Convert each improper fraction to a mixed fraction. (You will need extra pieces; combine sets with group members.)

5/3 5/2 9/4 11/3

b) Convert each mixed fraction to an improper fraction.

$2\frac{1}{3}$ $3\frac{1}{2}$ $1\frac{4}{5}$ $2\frac{2}{3}$

c) Are these common fractions in lowest terms? If not, simplify them.

6/8 9/12 10/16 5/7 7/2

d) Convert the following simplified binary fractions to simplified common fractions by finding a common denominator, then reducing if necessary. Use fraction strips to find a common denominator.

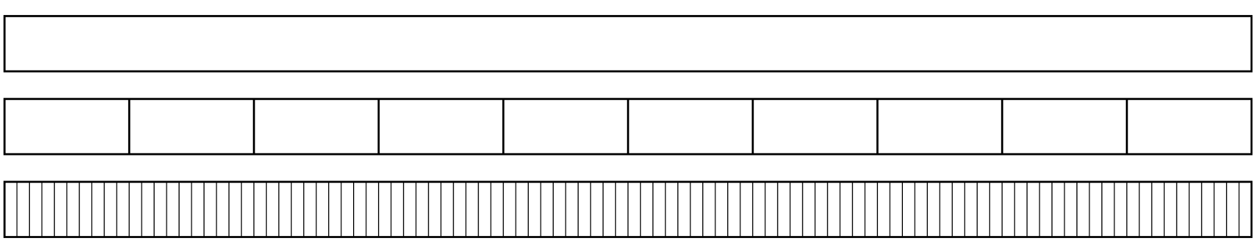
1/2 + 1/4 1/2 + 1/4 + 1/8 1/4 + 1/16

1/2 + 1/16 1/4 + 1/8 + 1/16 1/4 + 1/8

Do you see an easy way to find the smallest (lowest/least/smallest) common denominator for the parts of a simplified binary fraction?

E. A third system of fractions: decimal fractions

Definition: A *decimal fraction* is a fraction whose unit fraction parts are obtained by cutting the unit into 10 equal pieces, then cutting those pieces into 10 equal pieces, and so on.



[8] We are using the word "common" in two senses. A common fraction is an ordinary, fraction, as opposed to a decimal fraction (next section). Common denominators have their size in common, meaning the same.

Thus a decimal fraction uses pieces of size a tenth $\left(\frac{1}{10}\right)$, a tenth of a tenth (which is a hundredth, $\frac{1}{100}$), a tenth of a tenth of a tenth (a thousandth, $\frac{1}{1000}$), and so on. Here is a typical decimal fraction, which happens to be a mixed fraction: $14+\frac{3}{10}+\frac{7}{100}$.

In the modern world, for many reasons that will be explored in later sections, decimal fractions are used much more widely than any of the other forms of fractions we have discussed. We use a *place value* system instead of fraction notation, since it takes less writing and is much easier to type. In a place value. or *positional* system, the value of a number symbol depends on its place (position) in the number.

2	0	4	.	1	0	5
hundreds	tens	ones		tenths	hundredths	thousandths

The example above has places defined by blanks, with labels underneath them. In practice, labels up too much space and time, so the places are determined by finding the ones place, and figuring out the other places from there. The ones place is the one to the left of the decimal point, or if there is no decimal point, it's the rightmost place. Blank places in the middle of the number are not allowed, since it's hard to judge the size of a handwritten space. If there is a place in the middle with nothing in it, you have to fill in with 0 as a placeholder. This way of writing a number is called *decimal* form. Often numbers that have no whole number part will use a 0 in the ones place, just to make sure you notice the decimal point. Place value systems will be discussed in more detail in section 3.1.

Simplification of decimal fractions works the same as for other kinds: use as few and as large pieces as possible. However, the only pieces allowed are wholes, tenths, hundredths, and so on.

Discussion question 1. If a decimal fraction is written in decimal form, as in 0.105, is it already simplified? Why or why not?

Class activity 4. Decimal fractions in standard metric units.

So far, in this section, we have been sticking with a length model for the size of a number, and a single size for the main unit (that is, the "1"): the width of letter-size paper. Unfortunately, this makes hundredths too small to work with. For this activity, we'll use a standard unit of length that comes in decimal fractions: the meter.

Materials:

- Meter sticks or tape measures that have marks in both centimeters and millimeters. Have enough so that everybody has a chance to work with one, not just look.

- Cloth tape measures can be bought cheaply at craft stores, but make sure they have metric markings, not just inches. Metric metal retractable tape measures (the carpenter kind) can be found cheaply at dollar stores (their merchandise is mostly made in China.) It's hard to find low-cost metric tape measures at hardware stores in the U.S.

- A random number generator: calculators that have a random number function, or dice marked 0 through 9, or a spinner marked 0 through 9, or pieces of paper marked 0 through 9. Or you can just have people call out random digits 0 through 9.

  Take a random number of the form .abc (for example, .827): either get the calculator to give you one in that form, or pick the digits one at a time. Measure a length of that number of meters using the meter stick or tape measure. Note where 0 is on your stick. Make sure you do some examples that have a 0 in one or two of the places—-just make some up: .209, .004, .310. Practice until everyone can find the lengths easily.

F. Comparing fractions in two systems

Of the three fraction systems we have studied, some are easier for comparing the size of fractions.

Class activity 5. Comparing decimal fractions with length manipulatives

Materials:

Meter sticks or metric tape measures

a) Arrange each set of decimal fractions from smallest to largest, using a meter stick or metric tape measure. Some may be equal. You may not be able to find marks for some numbers—-they might be between marks.

1st set: 0.112, .212, .21, 0.021, .121

2nd set: 0.050, 0.500, 0.005, 0.505, 0.5, 0.05

3rd set: 0.372, 0.0327, 0.037, 0.2, 0.072, 0.0007, 0.30207

Alphabetical order is a way to arrange words in an order. Instead of "smaller" and "bigger", they are described as "before" and "after". In alphabetical order, you first compare the first letters. The letter that comes first in the alphabet determines which word comes first. If the first letters are the same, then you go on to the next letter, and so on.

You can use the same principle to order mathematical symbols (such as sums of fractions). This system is called *lexicographical order*.[9]

b) Find a non-physical (using the number symbols) method for comparing simplified decimal fractions written in fractional form, and adapt it for decimal form. Test your method by comparing lengths on the meter stick. Then try it out on these sets of fractions, ranking them from smallest to largest.

.2134, .1423, .2135, .2231, .4321

0.370, 3.007, .37, .307, 0.0037

0.11010, 1.00101, .0011111, 1.01001, .0101, 0.1, .01

Class activity 6. Comparing common fractions

a) Arrange these common fractions from smallest to largest, using fraction strips.

i) $\frac{7}{9}, \frac{6}{9}, \frac{2}{9}, \frac{4}{9}, \frac{8}{9}$

ii) $\frac{3}{7}, \frac{3}{4}, \frac{3}{8}, \frac{3}{10}, \frac{3}{2}$

iii) $\frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{6}{11}, \frac{3}{8}, \frac{5}{12}$

iv) $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}$

b) Which of the pair of fractions is bigger? Find the answer by trading pieces so that both fractions have the same denominator. Record your work by writing an inequality with fractions and the < sign.

i) $\frac{1}{2}, \frac{5}{8}$ ii) $\frac{1}{3}, \frac{1}{2}$ iii) $\frac{3}{4}, \frac{2}{3}$ iv) $\frac{3}{5}, \frac{1}{2}$

Comparing fractions using common denominators or numerators

Using fraction strips is easy to do, until you get to fractions that are so close that you need to worry about how carefully the strips were made. After using manipulatives in earlier grades, students need to use reasoning and computational methods for comparing fractions without resorting to manipulatives.

[9] "Lexicon" means dictionary.

Etty Wanda says:

Mr. Groener asked my class which fraction was bigger, $\frac{3}{7}$ or $\frac{4}{11}$. Some kids said that $\frac{4}{11}$ was bigger, because 4 pieces is more than 3 pieces. Other kids said $\frac{3}{7}$, because sevenths are bigger than elevenths. Mr. Groener said you have to find a common denominator. I asked, why can't you find a common numerator? Then the bell rang, so I missed getting sent to the principal's office.

Was anyone in this discussion right? Try both the common denominator and common numerator methods together with reasoning about the number and/or size of pieces. Which method seemed easier?

Example 2. Comparing 3/7 and 4/11 with different calculation methods

a) Common denominators:

$\frac{3}{7} = \frac{3 \cdot 11}{7 \cdot 11} = \frac{33}{77}$ (cut all the sevenths into 11 pieces, forming $\frac{1}{77}$s)

$\frac{4}{11} = \frac{4 \cdot 7}{11 \cdot 7} = \frac{28}{77}$ (cut all the elevenths into 7 pieces, forming $\frac{1}{77}$s)

Now all the pieces are the same size, and $\frac{3}{7}$ has more of them.

b) Common numerators:

$\frac{3}{7} = \frac{3 \cdot 4}{7 \cdot 4} = \frac{12}{28}$ (cut all the sevenths into 4 pieces, forming $\frac{1}{28}$s)

$\frac{4}{11} = \frac{4 \cdot 3}{11 \cdot 3} = \frac{12}{33}$ (cut all the elevenths into 3 pieces, forming $\frac{1}{33}$s)

Now both fractions have the same number of pieces (12 pieces.) $\frac{3}{7}$ has bigger pieces, so it is the larger fraction.

Comparing fractions with benchmark fractions

A *benchmark fraction* is a fraction that is easy to think about, and which other fractions are compared to. For example, $\frac{1}{2}$ is a benchmark fraction, and one way to roughly estimate the size of other fractions is to decide whether they are close to $\frac{1}{2}$, and more than $\frac{1}{2}$, or less than $\frac{1}{2}$. Other fractions usually considered as benchmarks are fourths, thirds, and sometimes tenths. Whether a fraction is a benchmark fraction depends on the context, and how easy it is for you to think about it.

Example 3. Which is bigger, $\frac{4}{7}$ or $\frac{5}{11}$?

Solution using benchmark fractions: $\frac{4}{7} > \frac{1}{2}$, because if the whole is split into 7 parts, 4 of these parts is more than half. $\frac{5}{11} < \frac{1}{2}$, because 5 out of 11 is less than half of the 11 parts. So $\frac{5}{11} < \frac{1}{2}$ and $\frac{1}{2} < \frac{4}{7}$ means $\frac{5}{11} < \frac{1}{2} < \frac{4}{7}$. This is a symbolic expression of the reasoning that the Measure Up first graders used: if A is less than B, and B is less than C, then you know that A is less than C. This is officially called the *transitive property of the "less than" relation.*

Comparing to benchmark fractions doesn't always work. In Etty Wanda's problem, comparing $\frac{3}{7}$ and $\frac{4}{11}$, both fractions are less than $\frac{1}{2}$ and greater than $\frac{1}{4}$. Maybe you can think of some other easier fractions to compare them to, but one of the other methods is probably less work.

G. Equivalent common fractions and rational numbers

Some rules emerge from trading pieces, as in the previous activity. Suppose you want to make halves into sixths. One half has the same length as 3 sixths; you could make sixths by cutting each half into three equal pieces. This makes 3 times as many parts in a whole (the denominator is 3 times as big), and there are 3 times as many parts in your length (the numerator is 3 times as big.)

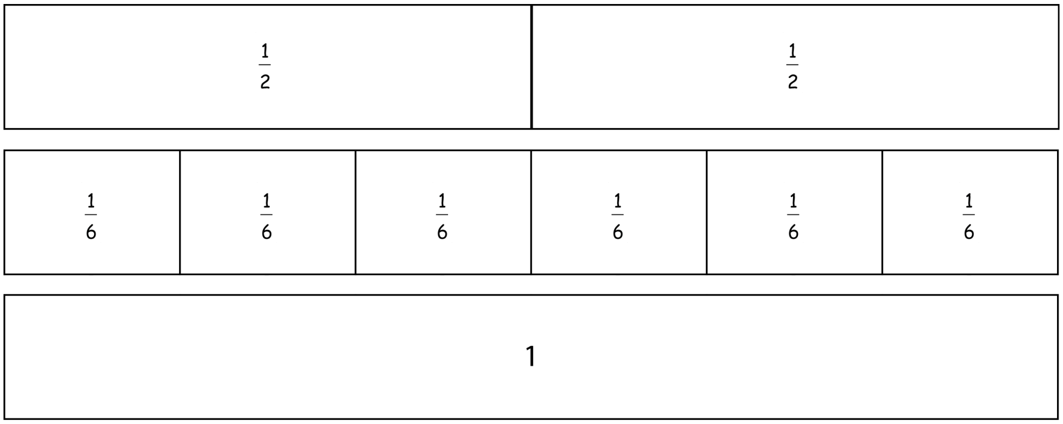


Figure 8. Each half is 3 sixths.

In symbols: $\frac{1}{2} = \frac{1 \cdot 3}{2 \cdot 3} = \frac{3}{6}$.

Class activity 7. Trade to make larger denominators/smaller pieces. Trade pieces to change the denominators. Write equations with fractions to record your work, as in the example above.

a) make 2 thirds into sixths

b) make 3 fourths into eighths

c) make 3 fourths into twelfths

Definition: *Equivalent fractions* are equal fractions that are expressed in different ways.

Usually the term "equivalent fractions" refers to common fractions. For example, $\frac{3}{6}$, $\frac{7}{14}$, $\frac{16}{32}$, $\frac{50}{100}$, and $\frac{1}{2}$ are all equivalent because they are different ways of expressing the same quantity, but using different sized fractional units.

Finding an equivalent fraction happens for various reasons. Sometimes it's simplifying a fraction. Sometimes it's making a fraction less simple, but so its denominator agrees with the denominator of another fraction. (Why is there no word for complicating a fraction, the opposite of simplifying?)

For each of the three systems of representing fractions discussed in this section,

- every fraction written in the system of fractions is equivalent to a simplified fraction in that system, and
- the simplified form is unique: there is only one way to write this simplified fraction in the system.

In other words, if two people simplify a fraction in different ways, but using the same system of fraction rules, they should end up with the same answer.

However, the simplified form of a fraction may be different in different systems. For example, $\frac{6}{16}$ has the following simplified forms in each of the three systems:

$\frac{3}{8}$	Common fraction
$\frac{1}{4}+\frac{1}{8}$	Binary fraction
$\frac{3}{10}+\frac{7}{100}+\frac{5}{1000}=.375$	Decimal fraction

Definition: A *rational number* is a combination (sum) of whole numbers and unit fractions, or any number that can be expressed that way.

Whole numbers are also rational numbers: it's just unnecessary to use any fractional units to express them. Rational numbers include negative whole and fractional numbers; we will discuss negative numbers in a later section. The way of combining the various units can be thought of either as putting physical pieces together, or as adding numbers.

This differs slightly from the usual definition given in modern textbooks: a rational number is any number that can be expressed as a simplified common fraction $\frac{a}{b}$. While these definitions are equivalent, the one given in this book is more flexible and more directly related to physical measurement.

Rational numbers on a number line

A child who has used fraction strips may think of the number line as made of chunks of lengths, end to end, and the number is the chunk, or the piece between two marks. This is not quite right: a number on a number line corresponds to one exact point/location, at the *end* of the last chunk counted.

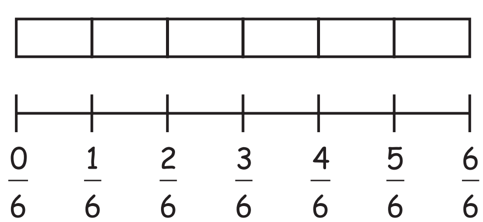
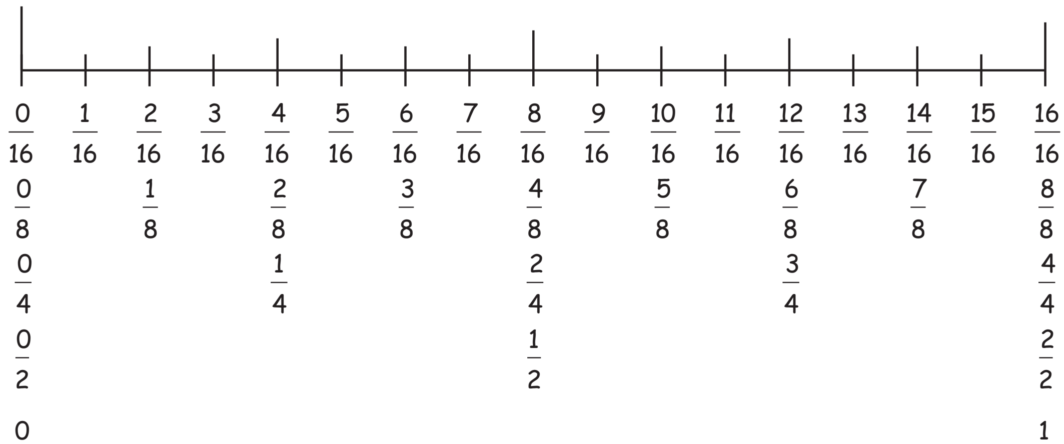


Figure 9. Fraction strips and a number line in sixths.

In the picture above, there are 7 marks on the number line: one for the beginning, and one for the end of each 1/6 fraction strip piece.

Here is a piece of a number line from 0 to 1, marked in sixteenths. Note that you get the eighths, fourths, and halves for free.



Class activity 8. Label the ruler fraction strips.

Label each uncut fraction strip as a ruler, as in the sixths and sixteenths examples above. Don't forget to label the left and right ends of the strip.

H. Problems, exercises, activities

1. Simplifying common fractions: coordinate physical and symbolic methods

Unsimplified —-Written numbers and fraction strips		Simplified—- Written numbers and fraction strips		Description of process
$\frac{2}{4} =$		$\frac{1}{2} =$		Trade 2 fourths for one half
$\frac{6}{8} =$				
$\frac{10}{4} =$				
$\frac{6}{3} =$				

2. Find a bunch of rulers, measuring tapes, measuring cups, thermometers, etc. that have scales marked with numbers. Figure out what unit, or what part of a unit the smallest

space between marks represents. Give both the number and the unit. (Example: 5 milliliters)

3. The fractional units in binary fractions are obtained by repeatedly halving.

a. How could you get 1/3 of a cake using halving? Act it out using a rectangle to represent the cake, and pretend that you are sharing it among 3 people. You may only cut pieces in half (horizontally or vertically). What happens? Write a sum of binary unit fractions that equals 1/3. (Note: young children sometimes think of this method.)

Use the same technique of halving to find and write binary representations of these common fractions. Rectangular cakes are easier to divide than circular ones.

b. Grandma has said you can have 2/5 of the pan of brownies she made. Use the same halving techniques as in part a to get your part of the pan.

c. You won a whole cake at the cake walk at the school carnival, but want to keep yourself from eating it all at once. You decide to eat it over the course of a week, with an equal part each day. Use the same halving techniques as in part a to cut it into 7 daily portions.

4. Art project: Figure out a way to color a square or rectangle that clearly shows the results of problem 3. In particular, the picture for 3.a should show clearly (without words) that the cake has been shared into 3 equal parts, and also clearly show the halves, fourths, etc. Suggestion: use graph paper to make a large square that can conveniently cut into halves repeatedly.

5. Can every common fraction be expressed as a *finite* combination of binary unit fractions? Give evidence for your answer; do problem 3 first.

6. Simplifying binary fractions with fraction strips.

a. Grab several binary fractions (using only the pieces $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$) and have your partner simplify the sum of those fractions using fraction strips. Record your work by writing equations with fractions, such as $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

b. After doing some of these, formulate a rule for recognizing when a binary fraction is simplified without using fraction strips.

c. Arrange these binary fractions from smallest to largest, using fraction strips. (Simplify if necessary before comparing.)

1/2 + 1/4	1/2 + 1/4 + 1/8	1/4 + 1/16
1/2 + 1/16	1/4 + 1/8 + 1/16	1/4 + 1/8

7. A short form for binary fractions.

Find a way to write simplified binary fractions in a decimal-like format, using place values and a dot. (You can't use the term "decimal point", because "decimal" refers to 10.) Try it on these fractions, and make up some more examples of your own:

a. 1/2 b. 1/4 c. 1/2 + 1/4 d. 1/4 + 1/8 + 1/16

e. Find a method for comparing binary fractions written in this decimal-like format. Give examples.

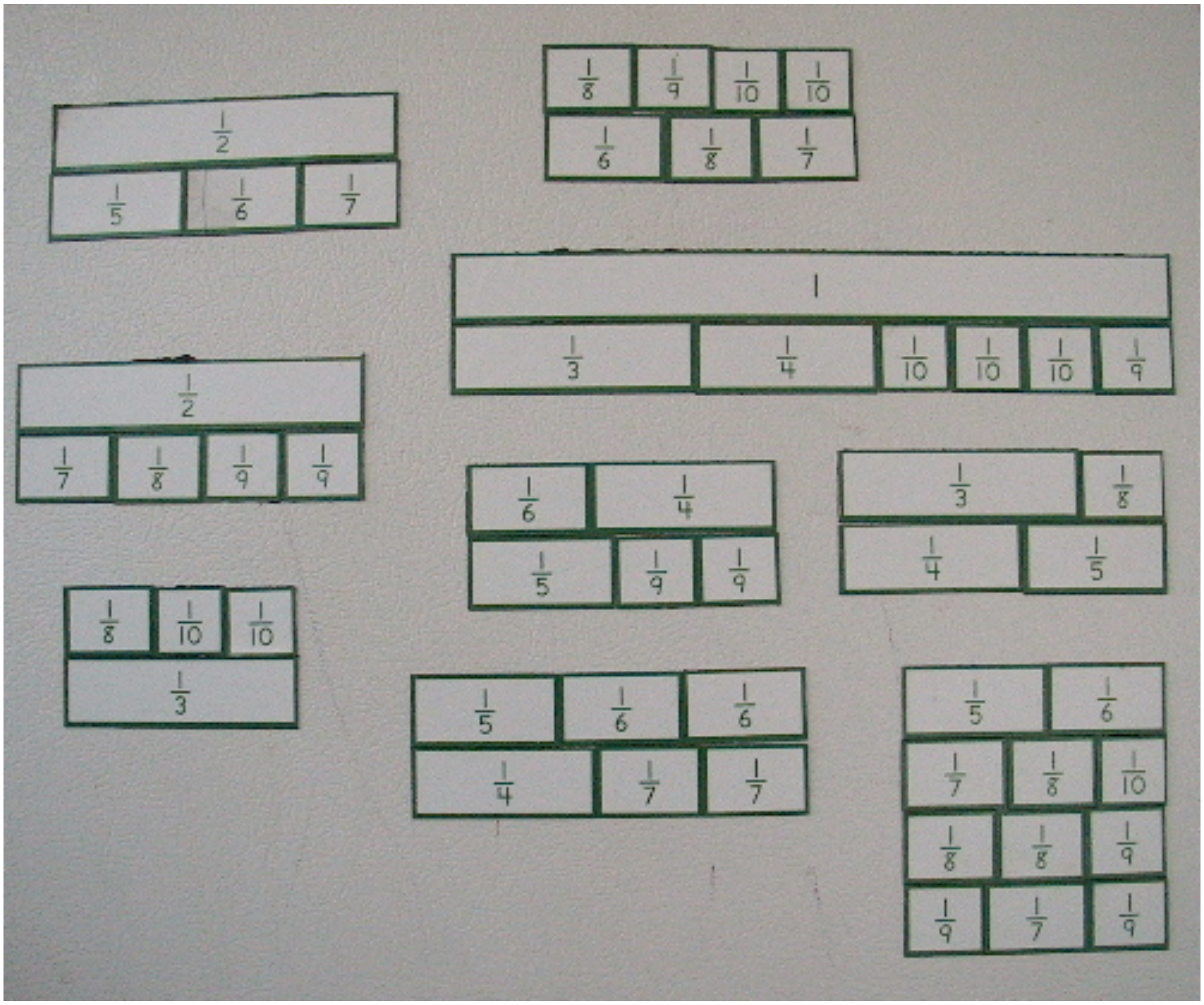
8. Use benchmark fractions or common numerators or denominators to compare these fractions. See if you can do the computations mentally.

a. $\frac{3}{7}, \frac{2}{7}$ b. $\frac{3}{5}, \frac{3}{9}$ c. $\frac{3}{7}, \frac{1}{2}$ d. $\frac{3}{7}, \frac{10}{21}$

e. $\frac{7}{9}, \frac{2}{3}$ f. $\frac{23}{28}, \frac{7}{10}$ g. $1\frac{3}{7}, 1.29$ h. $\frac{5}{3}, \frac{11}{7}$

i. $\frac{15}{37}, \frac{3}{7}$ j. $\frac{19}{32}, \frac{673}{1000}$ k. $\frac{21}{25}, \frac{6}{7}$ l. $\frac{13}{25}, \frac{17}{40}$

9. Here are some equations (?) of fractions using fraction strips. Are they really equal, or just close? How could you tell? Think about how you could reason mathematically about this .Try to use ideas that a 4th or 5th grader could use, and try to think of more than one method.



10. Use fraction strips to help answer these questions. U.S. measuring cups and spoons come in these sizes:

1 cup (c.), 1/2 c., 1/3 c., 1/4 c., 1 tablespoon (T), 1 teaspoon (t), 1/2 t, 1/4 t.

16 tablespoons = 1 cup, 3 teaspoons = 1 tablespoon

Figure out how to measure the volumes listed below using U.S. measuring cups and spoons, using the fewest number of steps (for example, filling one tablespoon 16 times is more steps than filling a cup once.)

a. 8 tablespoons
b. 7 tablespoons
c. 37 tablespoons
d. 6 teaspoons
e. 4 teaspoons
f. 20 teaspoons
g. 1/8 cup
h. 3/8 cup
i. 1/6 cup
j. 1/12 cup
k. 7/12 cup
l. 7/16 cup
m. 1/48 cup
n. 7/48 cup
o. 1/96 cup
p. What unit fractions of a cup can be measured exactly with U.S. measuring cups and spoons?

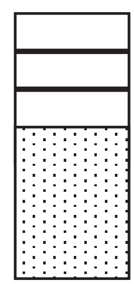
11. Evan's Italian pen pal had e-mailed him recipes for several fantastic desserts. However, the quantities were given in metric units, so Evan had to convert to U.S. cooking measures: cups, tablespoons, teaspoons. (See previous problem.) Unfortunately, the amounts didn't come out "even". Use fraction strips (or the number lines from problem 12) to figure out how to measure these decimal amounts using the measuring cups/spoons in problem 7.e. Some may not come out exactly equal. Say whether your answers are exact or approximate, and for approximate answers, give the amount of precision (such as "to within 1 teaspoon").

a. .375 cup b. 1.25 cups c. .33333 cup d. .47 cup

12. Project: Parallel number lines of fractions.

In this project, you will make a set of parallel fraction number lines that allow you to compare and convert fractions of different denominators at a glance.

Materials: 2 sheets of letter-size plain paper, or, preferably, card stock. (8.5 by 11 inches); your set of fraction strips rulers



a. Draw parallel lines across the paper for the number lines. Suggestion: use card stock with the sides aligned, then trace along the top.

b. Use the sixteenths ruler to mark the first line, and label the marks in sixteenths. Also label fourths and halves, as on the sixteenths number line in the text.

c. Use the twelfths ruler to mark and label the second line. Also label the equivalent fractions with smaller denominators whose marks are already on this line, such as 1/2. Some fractions will be on both the sixteenth and the twelfths number lines.

d. Continue with the next smallest unit fraction that isn't already on one of the lines, and include any other denominators you get for free. You should be able to get all denominators of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, and 16 on 6 different number lines.

Now you can use the other sheet of paper or card stock to accurately see which numbers line up. For example, $\frac{9}{16}$ on the sixteenths number line should align with $\frac{8}{12}$ on the twelfths number line, since they are both equivalent to $\frac{3}{4}$.

Use the number lines to answer the questions below. If the numerators are not whole numbers, say something like "between 3 and 4". Answer both in words and symbols like this:

Find a fraction in eighths equivalent to $\frac{1}{2}$. Ans: 4 eighths: $\frac{1}{2} = \frac{4}{8}$

How many fourths is 1/3? Ans: between 1 and 2 fourths: $\frac{1}{4} < \frac{1}{3} < \frac{2}{4}$ (and closer to 1/4 than to 1/2)

e. Find all (exact) equivalent fractions with denominators 2 through 12 and 16 to the given fraction on your lines. Some denominators may not have exact equivalents.

i. 1/2 ii. 3/4 iii. 2/3 iv. 3/12

f. How many tenths is each of the following fractions? There may not be an exact number of tenths.

i. 2/3 ii. 3/4 iii. 5/7 iv. 7/11

g. Suppose you needed to measure various lengths in decimal fractions of an inch, but you have only a standard ruler with sixteenths of an inch. Find the closest binary fraction mark(s) for these lengths. Note: your fraction strips are in tenths and sixteenths of a paperwidth. Will the same relationships hold between tenths and sixteenths of an inch?

i. 0.75 inch ii. 0.4 inch iii. 0.2 inch iv. 0.7 inch

h. Use a ruler marked in sixteenths to draw a line 5.7 inches long. (See the previous question for 0.7 inch.)

Math across the curriculum: social studies. Egyptian fractions

In ancient Egypt (see timeline in section 2.1), fractions were used by builders and for calculations of other quantities. For building, the main unit of length was the *cubit*. According to *How Many? A Dictionary of Units of Measurement* (see References for URL),

"The word *cubit* comes from the Latin *cubitum*, 'elbow,' because the unit represents the length of a man's forearm from his elbow to the tip of his outstretched middle finger. This distance tends to be about 18 inches or roughly 45 centimeters. In ancient times, the cubit was usually defined to equal 24 digits or 6 palms. The Egyptian royal or "long" cubit, however, was equal to 28 digits or 7 palms. In the English system, the digit is conventionally identified as 3/4 inch (width of a person's finger); this makes the ordinary cubit exactly 18 inches (45.72 centimeters)." The master cubit was carved in granite. The foreman of a building project was responsible for ensuring that the workers' cubit sticks were the same length as the master cubit.

Fractions of a digit were used for lengths smaller than a digit. On rulers found in tombs, the smallest fractions used were twelfths of a digit.

Measuring lengths with Egyptian fractions.

First, use the largest number of whole units that will fit in the length. Then, to measure the remaining length that is less than 1, find the biggest unit fraction that will fit into that length. Then find the next biggest unit fraction that will fit into the remaining length. Continue until you have the length exactly, or close enough for your purpose.

13. Measuring with Egyptian fractions using fraction strips. (Optional activity; complete set of fraction strips required.) Instead of using cubits, palms, digits, and fractions of a digit, we'll use paperwidths and fractions of a paperwidth.

Materials: For each person, one of each size of fraction piece in these sizes:

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}$

a. Measure the line segments at the end of this section. Some of them are designed to be an exact combination of the fraction strips, others are not. Remember: use at most one of each size, starting with the biggest one possible.

b. Then measure some objects with your fraction strips using the Egyptian fraction rules (at most one of each unit fraction piece, start with the biggest possible piece,…) For example, measure the length of letter size paper, the width or length of your desk or table, the length and width of a book, the length of a standard ruler, etc.

14. Comparing Egyptian fractions

a. The Egyptian fractions on the following list are guaranteed simplified. Use fraction strips to compare the simplified fractions. Then rearrange the list from largest to smallest. Some use some smaller pieces that you don't have. Either make a single copy, or use reasoning.

1/2 + 1/10	1/3 + 1/9	1/3 + 1/33	1/2 + 1/4 + 1/20
1/2 + 1/6	1/2 + 1/8	1/2 + 1/4	1/3 + 1/12
1/2 + 1/3	1/2 + 1/5 + 1/70	1/3 + 1/10	1/3 + 1/9 + 1/27

b. Formulate a rule for comparing simplified Egyptian fractions without using fraction strips.

15. (Extension) Here is a method for expressing any common fraction as a reduced Egyptian fraction. Assume your original fraction is a proper fraction $\frac{A}{B}$; if it's not, convert to a mixed fraction and just work on the fractional part.

a. Learn this procedure for generating an Egyptian fraction from a common fraction.

i. Divide B by A. If the result is a whole number, call it D. If not, round the quotient up to the next whole number, and call it D. That is the denominator of your first unit fraction. (Why is $\frac{1}{D}$ the largest unit fraction less than or equal to $\frac{A}{B}$?)

ii. Subtract $\frac{1}{D}$ from $\frac{A}{B}$ to get a new fraction: $\frac{A}{B}-\frac{1}{D}=\frac{A'}{B'}$. If $\frac{A'}{B'}=0$, you're done. (Why? What is the Egyptian fraction for your original fraction?) If not, repeat the process on the new fraction, $\frac{A'}{B'}$.

iii. Continue this process until you get 0 when you subtract. Then you can assemble the Egyptian fraction from all the denominators you got along the way.

b. Try the process on (at least) these common fractions: 5/6, 3/8, 9/10.

c. Prove that the procedure will always work by answering these questions.

i. In step a.i, why is $\frac{1}{D}$ the largest unit fraction less than or equal to $\frac{A}{B}$?

ii. In step a.ii, why are you done if you get 0? What is the Egyptian fraction for your original fraction?

iii. How can you be sure you will eventually get 0, rather than having the process continue infinitely?

I. References and Resources

Rowlett, Russ, *How Many? A Dictionary of Units of Measurement*, http://www.unc.edu/~rowlett/units

Segments to measure with fraction strips