4.3. Ratios

A. Definitions and examples of ratios

Recall these definitions from the section on division.

A *relation* is a correspondence between two sets of objects, numbers, or quantities; the relation can be given by a rule, description, or list that tells whether two things are related or not.

A *ratio relation* is a relation between numbers or quantities defined by division: two numbers A and B have a ratio of n, or n to 1, if, when you divide A by B, you always get the specific number n. This relation can be written several ways:

$$A \div B = n \quad \text{or} \quad \frac{A}{B} = n \quad \text{or} \quad A : B = n : 1$$

Order matters! A divided by B is not the same as B divided by A.

The word ratio can refer to the relationship, or to the number n.

Ratios behave in many ways like fractions.

Any ratio relating quantities A and B have this intuitive property: A is n times as big as B. When A is twice as big, B is twice as big. When A is r times as big, B is r times as big. (This includes r's less than 1, such as "half as big".) In other words, ratios can arise as comparisons by division (as opposed to comparisons by subtraction.)

Ratios can be classified into three types, as follows.

Part/whole ratios

Part/whole ratios are ratios of the same quantities and units. In this kind of ratio, you are comparing a part of some quantity to the entire quantity. In this situation, you use "per", or "for each", or "out of". Fractions are part/whole ratios, as are percents.

Example 1. Part/whole ratios.

a) There are 15 girls and 10 boys in a classroom. Thus there are 25 children in all. The ratio of girls to students is 15 to 25 (or 15:25), which simplifies to 3:5; that is, 3 out of every 5 students in the class are girls. This might also be expressed as a fraction: $\frac{3}{5}$ of the class is girls, or as a percent: 60% of the class is girls. For boys, the ratio is 10 boys : 25 students, or 2:5; $\frac{2}{5}$, or 40%, of the class is boys.

b) A punch recipe made entirely of liquids that makes a total of 16 cups of punch, calls for $1\frac{1}{2}$ cups of cranberry juice. The ratio of cranberry juice to punch is $1\frac{1}{2}$ cups to 16 cups, or 3:32 (the same ratio expressed with whole numbers); cranberry juice is $\frac{3}{32}$, or about 9%, of the punch.

c) A toy company puts out bags of marbles in four colors. People like red marbles the most, and black marbles the least, so each bag is packed so that for every black marble, there are 5 red marbles, 4 blue marbles, and 2 green marbles. You don't know how many marbles are in a bag, but you do know the part/whole ratios.

Color	**Red**	**Blue**	**Green**	**Black**	**Total**
(small bag)	5	4	2	1	12
(medium sized bag)	15	12	6	3	36
Ratio color to total	5:12	4:12	2:12	1:12	12:12
Ratio color to total, simplified	5:12	1:3	1:6	1:12	1:1
Fraction of bag	5/12	4/12	2/12	1/12	1
Percent of bag (approx.)	41.7%	33.3%	16.7%	8.3%	100%

Part/part ratios

Part/part ratios are another kind of ratios relating the same quantities and units. In this kind of ratio, you are comparing one part of some quantity to another part of the quantity; both are parts of the same whole. In this situation, you use "to", or the colon (:) notation; fraction and percent notation are not appropriate.

Example 2. Part/part ratios in Example 1.

a) There were 15 girls and 10 boys, so the ratio of girls to boys is 15 to 10 (or 15:10). It could also be simplified, like a fraction, to 3:2; that is, for every 2 boys, there are 3 girls. The order can be changed if you are careful with the units: 2 boys : 3 girls.

b) Suppose that someone has a different punch recipe consisting of various liquids, and they tell you that it uses $1\frac{1}{2}$ cups of cranberry juice and $2\frac{1}{2}$ cups of apple juice. The ratio of cranberry juice to apple juice is $1\frac{1}{2} : 2\frac{1}{2}$, or 3 : 5.

c) In the marble story, the ratio of colors red : blue : green : black is 5:4:2:1. This means that, for each black marble, there are 2 green marbles, 4 blue marbles, and 5 red marbles. (This is a part/part/part/part ratio!)

Most adults are accustomed to using part/whole ratios to describe ratio relationships. However, to some children, part/part ratios make more sense. Susan Lamon, a researcher who has studied multiplicative thinking in children, writes:

> "Although most mathematics curricula introduce ratios late in the elementary years, for some children, from the beginning of fraction instruction, the ratio interpretation of rational numbers is more natural than the part-whole comparison. These children identify [the two pictures below]

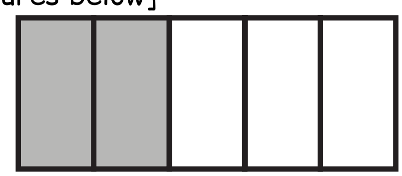
 and 

Figure 1. Part/part ratios 2 to 3

> as 2/3. There is some research evidence that when children prefer the ratio interpretation and classroom instruction builds on their intuitive knowledge of comparisons, they develop a richer understanding of rational numbers and employ proportional reasoning sooner than children whose curriculum used the part-whole comparison as the primary interpretation of rational numbers. These children favored discrete sets of coins or colored chips to represent ratios."

(Susan Lamon, *Teaching Fractions And Ratios For Understanding*; Lawrence Erlbaum Associates, 1999.)

Rates

Rates are ratios different quantities, as in speed: the ratio of distance to time. A rate can also be a ratio of two amounts of the same quantity, but not in a part/whole or part/part comparison. For instance, you could have a ratio comparing population of one town to the population of another town. A rate could also be the ratio of the same quantity in different units, as in converting units: inches to centimeters. This kind of ratio is very powerful, but more subtle to understand than the same-quantity type of ratio. In a rate, you can relate any quantity to any other quantity. In practice, the two quantities are usually related by being part of some situation, such as the distance a car travels and the time it takes.

Example 3. Rates

a) Grandma's lemon cake recipe uses 3 cups of sugar, and the modern healthier lemon cake recipe uses 2 cups of sugar. The ratio of sugar for Grandma's cake to the healthier cake is 3 to 2. Even though the same quantity is being measured, this is neither a part/part nor a part/whole ratio, since the amounts of sugar are parts of different cakes.

a) The toy company's marbles are packed 96 marbles per bag. Here you could make a ratio of number of marbles to number of bags. This is a rate because it relates two different quantities: number of marbles, and number of bags.

b) The marbles are packed 50 bags per case. (Ratio of bags to cases.)

c) The marbles are sold wholesale at $34 per case. (Ratio of dollars to cases.)

d) If you drive 48 miles in 1 hour, you are driving at an average speed of 48 miles per hour, or 48 mi./hr. (Ratio of distance to time.)

   If you drive 36 miles in 45 minutes (3/4 of an hour), you are also driving at an average speed of 48 mi./hr. This doesn't necessarily mean you were driving at exactly that speed the whole time; you could have been driving slower part of the time and faster part of the time. It also doesn't mean you drove for a whole hour, or 48 miles.

e) Converting units is a ratio relation between the same quantity measured in different units. It is a rate-style ratio because it is neither part/part nor part/whole.

Discussion question 1. This problem was on the California Standards Test for Grade 2:

David reads two pages every five minutes. How many pages will David have read after twenty-five minutes?

David's Reading

Minutes	5	10	15	20	25
Pages`	2	4	6	8	

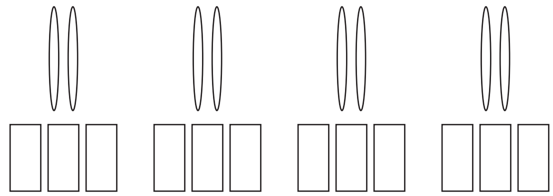
How is this problem related to the topic of this section? How might second graders solve the problem?

Representations of ratios.

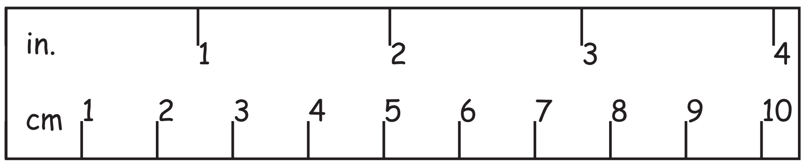
A *representation* of a mathematical concept is way of communicating it, for example, by a picture, symbol, table, or graph. Understanding and coordinating different mathematical representations is one of the themes of this course.

Pictures are sometimes a good way for young children to show a part/part or part/whole ratio, as in the 2/3 example in the quotation from Lamon above.

Here is a picture illustrating a rate: the ratio of pens to dollars if 2 pens cost $3.



Parallel number lines show the two quantities that correspond under some relation on separate number lines, marked with numbers aligned to show the relation. These were briefly discussed in the section on units. A good example to keep in mind is a ruler marked in inches and centimeters, where both scales start from the left end.



(If you look at your own ruler, you may find that the centimeters start from the right, and are upside down, so that when you rotate the ruler a half turn, the centimeters are on the top and start at the left.)

Tables list corresponding number across from each other in rows or columns. Methods that use tables are discussed in sections 0 and C below.

Graphs and charts. Function graphs, (the kind with a horizontal x and a vertical y axis) will be discussed at length in the next section. Pie charts (formally known as circle graphs) are a way to represent part/whole ratios. The circle represents the whole, and the "slices" represent all the parts. These will be treated in more detail in the section on angles.

Algebraic expressions and equations, along with algebraic methods for solving proportions, will be covered later in this section.

B. Additive methods for beginners

Generating ratio tables by adding

A *ratio table* is a row or column table with numbers for one of the quantities in the ratio relation in one row (or column) and numbers for the corresponding number for the other quantity in the other row (or column).

The units on the ruler shown above can be summarized in a ratio table in several ways.

A ratio table showing whole numbers of inches. Useful for converting inches to centimeters. Start with the fact that 1 inch is 2.54 inches. Arrows for the "add 1 inch" and "add 2.54 cm" functions show how the table was generated.

+1 +1 +1 +1 +1

in.	0	1	2	3	4	5
cm	0	2.54	5.08	7.64	10.16	12.70

+2.54 +2.54 +2.54 +2.54 +2.54

A ratio table showing whole numbers of centimeters. Useful for converting centimeters to inches. Start by finding the reciprocal relation: $\frac{1\text{ in}}{2.54\text{ cm}} = 0.39\frac{\text{in}}{\text{cm}} = \frac{0.39\text{ in}}{1\text{ cm}}$. Arrows for the "add .39 inch" and "add 1 cm" functions show how the table was generated.

+.39 +.39 +.39 +.39 +.39

in.	0	.39	.78	1.17	1.56	1.95
cm	0	1	2	3	4	5

+1 +1 +1 +1 +1

The entries in a ratio table don't need to be in a pattern, but the entries in the two rows or columns do have to correspond.

Parallel number lines and strip diagrams

A ruler where 0 inches and 0 centimeters are aligned is a pair of parallel number lines (the top and bottom edges of the ruler) that show the correspondence. Any two quantities that have a ratio relation can be shown on parallel number lines. It's easy to make them: just mark with equal spacing, and repeatedly add, as in the table, to get the labels.

Here is a way to represent the quantities in the pen situation by lengths, using Cuisenaire rods. (An online set of virtual Cuisenaire rods: http://arcytech.org/java/integers/integers.html).

White rod: Red rod: Light green rod:

The light green rods (the longest of the three sizes) represent pens; the red rods (the middle size) represent dollars. This configuration of rods shows that 2 pens cost $3.

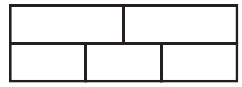
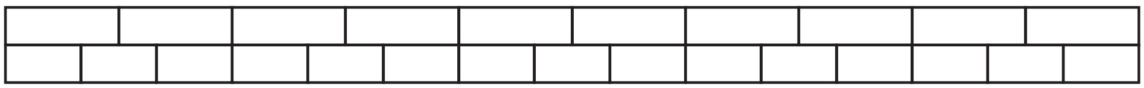


Figure 2.Top row: pens. Bottom row: dollars.

The configuration below shows prices for more pens. You can read off the cost for any number of pens up to 8.



Here is a ratio table that summarizes the picture.

Pens	2	4	6	8	10
Cost	$3	$6	$9	$12	$15

But what if you want 3 pens, or some other odd number? You can figure this out from the picture, too. 3 pens don't match a whole number of dollars, but you can fill in the space with a white rod, representing half a dollar.

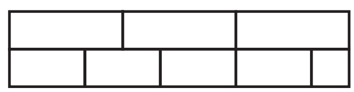
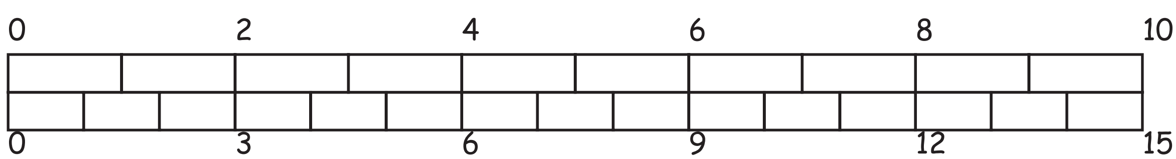


Figure 3. Three pens cost 4 and a half dollars.

Pens	1	2	3	4	5	6	7	8
Cost	$1.50	$3	$4.50	$6	$7.50	$9	$10.50	$12

The "trains" of Cuisenaire rods are really two parallel number lines. The top one is marked off in units of light green rods, and the bottom one is marked off in units of red rods.



By switching to number lines, we are implicitly starting to think in continuous, instead of discrete, quantities. That is, Cuisenaire rods are separate objects which are not ordinarily chopped into smaller pieces. But on a number line, you can measure off any length, including fractions between the whole numbers. The numbers on these number lines are measuring lengths, not just counting blocks. In the pens example, it doesn't make sense to have fractions of pens, but it *is* necessary to think about fractional dollars.

The elementary math curriculum in Singapore, which is usually highest in the world in international math education studies, uses "strip diagrams" for solving problems before introducing algebra. They look like the Cuisenaire diagrams in that numbers are represented as the lengths of strips. However, the numbers could be anything, even unknown numbers.

While repeated addition is an easy, elementary way to get a ratio relation, it has the shortcoming that it's harder to find values that are in between two numbers on the table or number line. The mathematical term for figuring out values in between two known values in a table is *interpolation*.[1] Here's an example that illustrates the thinking needed to interpolate a ratio table.

Example 4. Ellen is in charge of the after-school art club. She has bought 30 yards of ribbon for a craft project; it cost \$24. The children have brought money to buy pieces of ribbon. Make parallel number lines that allow you to read off the answers to questions such as these:

Ben wants 3 yards of ribbon. How much should he pay?

Amalia's mother gave her a \$5 bill. How much ribbon can she get?

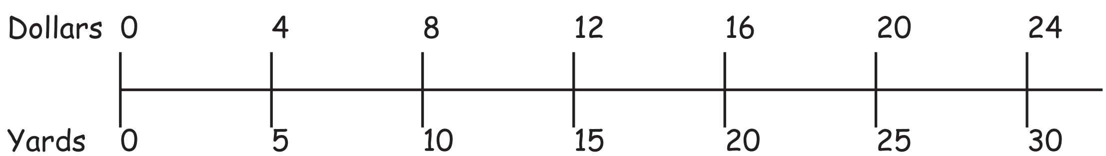
The ratio of cost, in dollars, to yards is

$$\frac{24\ \$}{30\text{ yd.}} = \frac{24}{30}\,\frac{\$}{\text{yd.}} = \frac{4\ \$}{5\text{ yd.}}$$

That is, \$24 for 30 yards is \$4 for 5 yards, and these are the smallest whole numbers of dollars and yards that express this ratio. Using these numbers and addition gives this ratio table:

Cost (\$)	0	4	8	12	16	20	24
Yards	0	5	10	15	20	25	30

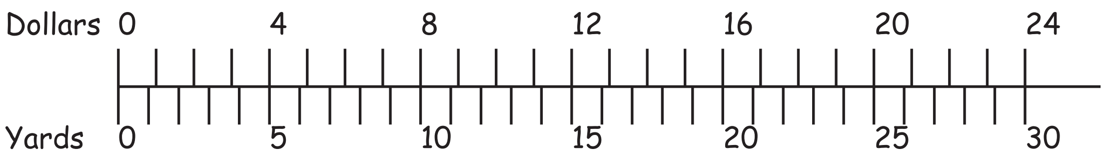
The number lines below show how cost corresponds to amounts of ribbon, but they don't show quite enough detail to answer the questions: not all dollar amounts or yardages are shown. Ellen can estimate: Ben's 3 yards of ribbon will cost between \$0 and \$4, closer to \$4, but they both probably want a better answer.



To get whole numbers of dollars, you have to split the distances between marks on the upper number line into fourths.

To get whole numbers of yards, you have to split the distances between marks on the lower number line into fifths. These marks do not coincide with the in-between dollar marks.

[1] Literally translated, this means "to polish in between", smoothing out the gaps in the table.



This is more useful. To find how much Ben should pay for 3 yards of ribbon, find the mark for 3 on the Yards line, read up, and figure out where it comes out on the Dollars line. It looks like 3 yards costs between 2 and 3 dollars; $2.50 might be a reasonable price. Amalia, with her $5, can buy a little over 6 yards of ribbon.

What if Ellen and the children are concerned about the prices, and want to make sure nobody is over- or under-charged? This diagram still isn't detailed enough. More work with ratio tables might help.

The simplest ratio to work with seems to be 4 dollars to 5 yards. Separating the numbers from the units, we get

$$\frac{4}{5}\frac{\$}{\text{yd.}}=\frac{\$.80}{\text{yd.}}$$

(since 4/5 of a dollar is 80 cents.) This reads: 80 cents per yard, or 80 cents for one yard.

The reciprocal will give the number of yards for 1 dollar:

$$\frac{5}{4}\frac{\text{yd.}}{\$}=1\frac{1}{4}\frac{\text{yd.}}{\$}$$

That is, for $1 you can get $1\frac{1}{4}$ yards of ribbon, and 1 yard of ribbon costs $\frac{4}{5}$ of a dollar, which is $.80. Here is a ratio table with whole numbers of dollars in the first part, and whole numbers of yards in the second part.

Dollars	$1.00	$2.00	$3.00	$4.00	$0.80	$1.60	$2.40	$3.20	$4.00
Yards	1 1/4	2 1/2	3 3/4	5	1	2	3	4	5

They might want a ratio table for even smaller amounts than a dollar or a yard. See the problems for further practice.

C. Multiplicative methods for working with ratios

Rules for working with ratio tables

Discussion question 2. A class of 35 (and 3 teachers/parents) ate 14 quarts of chili at their cookout, how much should we plan for the whole school (450 students and 22 teachers/staff/parents)?

Here is work typical of students who don't yet know how to do proportions, and some who do. (The last 3 columns are not typical; they are included to illustrate the mathematical process of finding an exact solution.) See if you can figure out what is being done at each step before reading on.

People	38	380	76	456	19	475	1	3	472
Qt. of chili	14	140	28	168	7	175	$\frac{7}{19}$	$1\frac{2}{19}$	$173\frac{17}{19}$
Step	1	2	3	4	5	6	7	8	9

Reasoning behind the steps:

Step 1: Known information

Step 2: 10 times as many people and 10 times as much chili is closer to 483

Step 3: 483-380=103 people left to feed, so double 38 and 14: 76 and 28

Step 4: Add the people fed so far and the chili: 456 people, 168 qt. of chili.

Step 5: Try half of 38 and 14: 19 and 7.

Step 4: Add the people fed so far: 475 people, 175 qt. of chili. Most people would stop here, and would probably even put in more chili to make sure there's enough. But this is a math problem, so we'll go for the (unrealistically precise) exact answer. We have 3 too many people

Step 7: 1 (average) person will eat $\frac{7}{19}$ of a quart of chili, so

Step 8: 3 people will eat $\frac{21}{19} = 1\frac{2}{19}$ qt. of chili.

Step 9: Subtract the 3 people and their chili for the exact amount for 472 people. It's a bit less than 175 qt.

Distilling this example into a set of rules for making new columns in a ratio table, we have:

I. Add one number to one row and a related number to the other row. The numbers to add are the ratio, expressed in the "A to B" format. Add or subtract any two (or more) columns. This includes adding a row to itself.

In operations of type I, the units do not change: in the chili example, people + people = people, and quarts of chili + quarts of chili = quarts of chili.

Applied to the 2 pens for $3 example, this rule can generate the pens/dollars table additively. The rule can also generate larger entries by adding columns. However, it can't generate smaller, in-between quantities.

+2 pens +2 pens +2 pens

pens	0	2	4	6	8	14
dollars	0	3	6	9	12	21
		$+\$3$	$+\$3$	$+\$3$	add col. for 4 pens to itself	add cols. for 6 and 8 pens

II. Multiply or divide the entries of a column by the same number.

This is the stretching/shrinking concept of multiplication: twice as much, half as much, etc. The units do not change: 10 times 38 people is 380 people, and they eat 10 times as much chili. Here a Type II operation is applied to the pens/dollars table to quickly figure out the cost for a large number of pens.

x2 x10 x47.5

pens	2	4	40	1900
dollars	3	6	60	2850

x2 x10 x47.5

In practice, people find it easier to multiply and divide by 10 and by 2, and to break a procedure into many smaller steps. However, you could have done the chili problem in fewer steps, using daisy chain ideas:

People	38	1	472
Qt. of chili	14	$\frac{7}{19}$	$173\frac{17}{19}$
Step	1	2	3

Step 2: Find the amount of chili for one person. (Shrink by 1/38)

Step 3: Find the amount for 472 people by multiplying the amount for one person by 472.

Another way to express this sequence of computations is using dimensional analysis.

$$\frac{14 \text{ qt.}}{38 \text{ persons}} = \frac{7}{19}\frac{\text{qt.}}{\text{person}} = \frac{\frac{7}{19}\text{ qt.} \times 472}{1 \text{ person} \times 472} = \frac{173\frac{17}{19}\text{ qt.}}{472 \text{ persons}}$$

In words: 14 quarts for 38 people is 7/19 of a quart per person, which makes $173\frac{17}{19}$ quarts for 472 people.

The daisy chain could be simplified to a single step: multiply by $\frac{472}{38}$, the one-step way to get from 38 people to 472 people.

A third way to work in ratio tables is a little less intuitive, and related to thinking about units and conversions. In the pens story, we could think of this as a conversion of number of pens to number of dollars. In the chili story, it's a conversion from people to quarts of chili. Here's a calculation with units:

$$10 \text{ pens} \times \frac{\$3}{2 \text{ pens}} = \$\frac{10\times 3}{2} = \$15.$$

In words: 10 pens, at \$3 per 2 pens, is a cost of \$15.

To go from pens to dollars, multiply by $\frac{\$3}{2 \text{ pens}}$, which is the same as $\frac{3}{2}\frac{\$}{\text{pen}} = 1.5\frac{\$}{\text{pen}}$, or \$1.50 per pen.

Pens	2	1	1900	
Dollars	3			84

$\times\frac{3}{2} = 1.5$ $\times\frac{2}{3}$

To go from pens to dollars, multiply by $\frac{\$3}{2 \text{ pens}}$, which is the ratio in the table, expressed as a number. (That is, if you're checking for a ratio relation, this is the number you always get when you divide dollars by pens.) To go from dollars to pens, multiply by the reciprocal, $\frac{2 \text{ pens}}{\$3}$, which is the same ratio in the opposite order (pens to dollars). In the example above, the "multiply down" arrow is useful to find the cost of 1900 pens. The "multiply up" arrow is useful to find how many pens you can buy for \$84.

Here's the chili problem again, done as a multiplication with units.

$$472 \text{ people} \times \frac{14 \text{ qt. of chili}}{38 \text{ people}} = \boxed{} \text{ qt. of chili}$$

(To get the number part of the answer by multiply 472 by 14, then divide by 38.)

If you need to know the amounts chili for a number of different sized gatherings, this would be the most efficient way to go: simply multiply the number of people by $\frac{14}{38}$ quarts per person. (This is assuming people eat similar amounts to your class.)

Event	**Class cookout**	**Family reunion (lots of kids)**	**Boy scout camp**	**Leftovers**
People	38	63	105	
Qt. of chili	14			5

$\times\frac{14}{38}$ $\times\frac{38}{14}$

If you have 5 quarts of chili left over, you can figure out how many people you should invite for supper to finish it off by using the up arrow.

Here's a summary of rules for ratio tables:

I. Add across. Add any column in the table to any other column, including itself. This column can be thought of as *the* ratio you're working in, expressed in the two-number format: A to B.

II. Multiply/divide across. Multiply a column by any number.

III. Multiply/divide up/down. Multiply by the ratio, expressed as a number, to go up or down. If you know the number to go one direction, the arrow in the opposite direction is its reciprocal.

Ratio boxes

A *ratio box* is just a ratio table with two rows and two columns. It's the minimum amount of information you need to solve a ratio problem. A ratio box can also be considered to be a grid table with two rows and two columns. It's often useful to write in the headings in the table; make sure you use careful descriptions to keep the four numbers in the right places.

Generally you use one of the multiplication rules (II or III) to get to the unknown number. You might need to do a daisy chain to figure out what to multiply by. Here's the pens example again:

	small amount	**large amount**
pens	2	1900
dollars	3	?

$\times\frac{3}{2}$

$\times 950$

If you don't mind dealing with fractions, use the down arrow. If you don't mind dealing with big numbers, use the across arrow. In either case, you get the same answer:

$$3\times\frac{1900}{2}=3\times\frac{1}{2}\times 1900=\frac{3}{2}\times 1900=2850$$

These expressions for the answer are made up of three parts: multiply by 3, multiply by 1900, divide by 2, but grouped in different ways (the associative property.)

Ratio boxes have another special property: if you multiply across both diagonals, the products are equal. Most people will be convinced by examples. Check that it's true in the four ratio boxes below.

2	4
3	6

$2\times 6=3\times 4$

5	7
10	14

$5\times 14=10\times 7$

1.2	100
3.6	300

$1.2\times 300=3.6\times 100$

1/3	4/5
3/2	18/5

$\frac{1}{3}\times\frac{18}{5}=\frac{3}{2}\times\frac{4}{5}$

In general, a ratio box with any entries A, B, C, D, as below, satisfies $A\times D=C\times B$.

A	B
C	D

Here's a proof that uses the "multiply across" rule of ratio tables. Extend the ratio box to a ratio table, to leave room for new columns, all of which have the same ratio.

x D x C

A	AD	B	CB
C	CD	D	CD

x D x C

First multiply the first column (A and C) by D, top and bottom, to obtain AD and CD. Next, multiply the B, D column by C, top and bottom, to obtain CB and CD. The second and fourth columns are equal in the bottom position, so they must be equal in the top position: that is, AD=CB.

Example 5. Solving problems with ratio boxes and tables.

a) Scaling an image. Suppose you have a photo that is 12 cm wide and 16 cm high, and you want it to be 40 cm high. How wide will it be? What scale factor should you use, both as a number and as a percent?

	original	scaled copy
width	12 cm	
height	16 cm	40 cm

$\times \frac{40}{16}$

Figure out how to multiply to go right using a daisy chain for the height row: by $\frac{40}{16} = \frac{5}{2}$.

Use the same multiplier for the width row: multiply the width by $\frac{5}{2}$; the width on the copy is $12 \times \frac{5}{2} = 30$ cm. The scale factor is $\frac{5}{2} = 2.5 = 250\%$. In this problem, we needed to know the right arrow for the scale factor, so using the down arrow or cross multiplying would be extra work.

b) Scaling a recipe. A recipe that serves 2 uses $2\frac{1}{2}$ tsp. of chile. How much chile is needed to serve 3?

	original recipe	scaled recipe
servings	2	3
tsp. of chile	$2\frac{1}{2}$	

$\times \frac{2\frac{1}{2}}{2}$

$\times \frac{3}{2}$

Method 1: The right arrow that goes from 2 to 3 is multiplication by $\frac{3}{2}$. Fill in the box with $2\frac{1}{2} \times \frac{3}{2} = \frac{5}{2} \times \frac{3}{2} = \frac{15}{4} = 3\frac{3}{4}$ tsp. of chile.

Method 2: The down arrow that goes from 2 to $2\frac{1}{2}$ is multiplication by $\frac{2\frac{1}{2}}{2} = 2\frac{1}{2} \times \frac{1}{2} = \frac{5}{2} \times \frac{1}{2} = \frac{5}{4}$. Fill in the box with $3 \times \frac{5}{4} = \frac{15}{4} = 3\frac{3}{4}$ tsp. of chile.

Method 3: Call the number in the empty box C. Cross multiply to get the equation $2 \cdot C = 2\frac{1}{2} \cdot 3$. Multiply both sides by $\frac{1}{2}$ and simplify: $C = 2\frac{1}{2} \cdot 3 \cdot \frac{1}{2} = \frac{5}{2} \cdot 3 \cdot \frac{1}{2} = \frac{15}{4} = 3\frac{3}{4}$ tsp. of chile.

All three methods involved a similar amount of fraction multiplication, and gave the same answer. The easiest method is a matter of personal preference.

c) Changing units. Some real estate listings give the area of a piece of land in square feet; others give it in acres. Set up a table for converting back and forth.

This ratio table is set up in columns because the words fit better that way. To go between the columns, multiply by the ratio 43,560 sq. ft. to 1 acre, expressed as a number in the appropriate order. Note that the reciprocal $\frac{1}{43560} \approx .000023$

Address	**Lot area, sq. ft.**	$\xrightarrow{\times .000023}$ $\xleftarrow[\times 43,560]{}$	**Lot area, acres**
1288 Mountain Dr.	13,000	$\xrightarrow{\times .000023}$	
398 5th St.	8712	$\xrightarrow{\times .000023}$	
6002 Texas Av.		$\xleftarrow{\times 43,560}$	.42

Fill in the table yourself.

D. Algebraic methods for working with ratios

In traditional books, solving ratio problems is done using proportions. Later, proportions are treated as algebraic equations to solve.

Definition: A *proportion*[2] is an equation of two ratios.

In (pre-)algebra, students are taught to solve proportions by techniques that will later be used to solve more complicated equations. Here is an example.

Example 6. Fran bought 1.7 pounds of fresh brussels sprouts for $3.04. How much will 2. 5 pounds of brussels sprouts cost?

A labeled ratio box might help in organizing the information.

[2] In common language, "proportion" sometimes means just a ratio, as in the proportions of a building or person, or the ratio of water to lemonade powder.

	Fran's purchase	bigger amount
weight (pounds)	1.7	2.5
cost (dollars)	3.04	x

There are four different ratios you could use in this problem: weight to cost, cost to weight, bigger amount to Fran's amount, or Fran's amount to the bigger amount. Each ratio gives a different way of setting up the proportion.

Method 1: Weight to cost. The ratio of weight to cost for Fran's sprouts is 1.7 to 3.04, or $\frac{1.7}{3.04}$. The units for this ratio are pounds per dollar. This ratio is the same for any purchase of brussels sprouts, so it's the same as $\frac{2.5}{x}$. The proportion is

$$\frac{1.7}{3.04} = \frac{2.5}{x}$$

One way to solve this equation algebraically is to multiply both sides by 3.04x, since that will cancel both denominators. (Or you could cross multiply to get the same equation.)

$$\begin{aligned} 3.04x \cdot \frac{1.7}{3.04} &= \frac{2.5}{x} \cdot 3.04x \\ x \cdot 1.7 &= 2.5 \cdot 3.04 \\ x &= \frac{2.5 \cdot 3.04}{1.7} = 4.47 \end{aligned}$$

So 2.5 pounds of sprouts cost $4.47.

Method 2: Use cost to weight (unit cost). The proportion you get is $\frac{3.04}{1.7} = \frac{x}{2.5}$

In this case multiplying both sides by 2.5 will give $x = \frac{2.5 \cdot 3.04}{1.7} = 4.47$,

as in the previous method.

Method 3: Ratios of bigger amount to Fran's amount. The proportion is:

$$\frac{2.5}{1.7} = \frac{x}{3.04}$$

Dividing the left side, 2.5 divided by 1.7, says that the bigger amount is 1.47 times as much as Fran's amount, so it will also cost 1.47 times as much as Fran's cost. Solving this equation gives exactly the same multiplication and division and answer as before.

Method 4: Ratio of Fran's amount to bigger amount. The proportion is:

$$\frac{1.7}{2.5}=\frac{3.04}{x}$$

This equation says that the weight of Fran's sprouts is $\frac{1.7}{2.5}=0.68$ times as big as the other amount (68%), so it will cost 0.68 as much. The algebra of solving this equation again involves multiplying 2.5 by 3.04, then dividing by 1.7.

Indirect measurement using ratios

Some measurements in real life are made directly, but many are made by measuring something that's easier to measure, then using a ratio to compute the quantity you actually want to know.

Example 7. Susan was at a conference in another city, and went for a run near her hotel. She knew from experience that she was running at a pace of 9 minutes per mile. She ran for 30 minutes. How many miles did she run?

In this situation, it's difficult to measure the distance of her run, but easy to measure the time (with a watch.) The ratio for her pace, 9 minutes per mile, gives a proportion for the distance of her run. You could also think of it as changing units with dimensional analysis.

$\frac{9 \text{ min.}}{1 \text{ mile}}=\frac{30 \text{ min.}}{x \text{ miles}}$, so $9x=30$, giving $x=\frac{30}{9}=3\frac{1}{3}$ miles.

See the Weigh the Car project in the problems for another indirect measurement. Think up some of your own!

E. Problems, exercises, more activities

1. Which of the following are ratio relations? How did you decide? Make up a longer story that explains the numbers.

a. Joe's distance (D) and time (T) in running a marathon (26 miles):

D (miles)	1	2	3	4	13	20	26
T (minutes)	7	14	21	29	101	173	245

b. Prices as Whataburger, on "2 Fer" Wednesdays

B (number of burgers)	2	4	6	10	20	30
C (cost, in dollars)	3.50	7.00	10.50	17.50	35.00	52.50

c. Prices at the New York Bagel Deli

B (number of bagels)	1	2	3	6	13	26	144
C (cost, in dollars)	.89	1.78	2.67	5.19	10.38	20.00	93.60

d. Area (A) of squares and length of side (s)

A (sq. cm)	1	4	25	9.61	23,409
s (cm)	1	2	5	3.1	153

e. Perimters (P) of squares and length of side (s)

P (sq. cm)	4	8	20	12.4	612
s (cm)	1	2	5	3.1	153

2. Make a Cuisenaire rod diagram that shows the answer to each problem.

a. A good rule for making vinaigrette salad dressing is 2 parts oil to 1 part vinegar. (Add other seasonings to taste.) If you only have 5 tablespoons of vinegar, how much oil should you use?

b. Instant noodles are on sale, 3 packages for a dollar. How much will 10 packages cost? What is the unit cost (dollars per package)?

c. David reads two pages every five minutes. How many pages will David have read after twenty-five minutes? How fast is he reading, in pages per minute? How many minutes does he take to read one page?

d. Earlene earns $300 each week taking care of an elderly neighbor; she works a bit every day. How many days will it take her to earn $500? What is her pay rate, in dollars per day?

3. Make a ratio tables and corresponding parallel number lines for the dollars/yards of ribbon relation in Example 4. ($4 for 5 yards of ribbon.) Use these ranges and units:

a. From $0 to $1 in increments of 20 cents; measure ribbon in inches and fractions of an inch

b. From $0 to $.20 in increments of 5 cents; measure ribbon in inches and fractions of an inch

c. From 0 inches to 12 inches; measure money in cents, rounded to the nearest cent.Make ratio tables for conversions from feet to meters.

a. Make a table of whole numbers of feet from 0 to 10 feet.Make a table of whole numbers of meters from 0 to 10 meters.

5. Jamal earns $5 an hour for doing chores.

a. Make a ratio table and corresponding parallel number lines that show how much money he can make for working up to 2 hours. Include increments of 10 minutes on the time scale and increments of $.25 on the money scale.

Answer two ways: using your number lines, and using a numerical method of your choice. Your number line answer may be an estimate. Use the numerical method to check.

b. Jamal needs $8.75 to go to the movies. How long does he have to work?

c. If he works for 4 hours and 35 minutes, how much money will he make?

6. The amusement parks below have different pricing schemes for multi-day tickets.

a. Find the cost per day for each park for each number of days in the table.

b. Which, if any, of the following pricing schemes for amusement parks is a ratio relation? How is this related to part a?

c. For each park, is it cheaper to buy a separate ticket each day, or to buy a multi-day pass? Does it make a difference?

Days at Big Thrills	1	2	3	4	5	6	7
Cost	$37.95	$75.90	$113.85	$151.80	$189.75	$227.70	$265.65

Days at Wild Ride	1	2	3	4	5	6	7
Cost	$43.25	$84.25	$125.25	$166.25	$207.25	$248.25	$289.25

Days at KaZamm	1	2	3	4	5	6	7
Cost	$44.75	$87.50	$128.25	$167.00	$203.75	$238.50	$271.25

7. Find one or more ratios in each situation, and express each as a number (not with "to" or ":"). Give units, such as miles per gallon. Is there a name for the ratio? (For example, mileage is the name for the ratio of miles to gallons.)

a. Albertson's is selling a 24-pack of 12-ounce cans of soda for $5.99.

b. Karen drove the 23 miles from Riverside to her friend's house at rush hour, and it took 45 minutes.

c. Mark's cell phone plan costs $35 each month for 600 minutes. If he goes over, it costs $5 for another 100 minutes.

d. Alexander offered to work for 120 hours on his father's web pages if his father would buy him a digital camera for $1000.

e. A cheetah can go from 0 to 60 miles per hour in 4 seconds.

8. Look in a newspaper or news magazine (most have internet versions) and find three different ratios. List what they are, summarize the context in a sentence or two, and give the name of the article, date, and publication. Give the URL (http://…) if you found it on the internet.

9. Practice using ratio table techniques. Solve each problem in four ways:

Using ratio table operations of types I and II. (Pretend you're a kid and don't know how to do proportions yet.)
Using ratio table operations of type III
Using a ratio box
Using a proportion, then doing algebra

Some of the methods may be almost identical.

a. The ratio of boys to girls at the Boys and Girls Club is (approximately) 8 to 5. There are 67 boys. How many girls are there?

b. Every time Yoshi goes to help his aunt, she gives him $10, and his parents give him another $3 to encourage his work ethic. How many times will he have to go to earn $75? How much will his aunt have paid in all? How much will his parents have paid?

c. A recipe calls for $3\frac{1}{2}$ cups of kidney beans and $2\frac{1}{2}$ cups of green beans. If you have 4 cups of green beans to use up, how many cups of kidney beans will you need?

d. 56% of the students at Murphy school are on free/reduced price lunch. The school lunch supervisor tells you that 189 students had the free/reduced lunch today. Estimate the number of students at the school.

10. Use a ratio box to solve this problem. A demonstration covers one block of a street in a city. The block is 1/12 of a mile long and the street is 30 feet wide. You experiment with the class, and find that 12 people can pack into a space 10 feet by 5 feet and have enough room to yell and wave signs. Estimate the number of people at the demonstration.

11. Use ratio tables to organize your work. Include headings.

Maribel's teaching job pays $45,000 a year. There are 180 days of teaching, 6 hours a day, 5 days a week. Maribel estimates that she spends 4 hours for each teaching day preparing and grading.

a. What is her hourly rate for work at school?

b. What is her hourly rate for *all* her work, including preparation and grading?

c. She gets a grant for summer work that pays her the hourly rate she gets for working during the school year (just the time in school, not the preparation at home). If she works for 5 weeks, 40 hours per week, how much will she earn?

12. 16 tablespoons (T.) = 1 cup; 3 teaspoons (tsp.) = 1 tablespoon. If you make only 1/3 of a cake recipe that requires $1\frac{3}{4}$ cup of cocoa, how much cocoa will you need? Give measurements in a combination of sizes for which there are measuring cups and spoons: $\frac{1}{2}$ cups, 1/3 cups, $\frac{1}{4}$ cups, tablespoons, teaspoons. Use the fewest number of measurements possible, and round off to the nearest teaspoon if necessary.

13. Project: Be a human odometer and pedometer, part 1. We'll do part of this in class.

The odometer on your car measures the distance you have traveled. A pedometer is a small device you wear that counts your steps and figures out how far you have walked.

Materials: A watch with a second hand or a stopwatch, paper and pencil (and maybe a partner), a long tape measure, or a hallway with tiles.

a. What to do: Measure a distance in meters to walk using a measuring tape, or go to a track at a school. Decide on a normal walking pace that you could keep up for quite a while. Walk the distance, counting your steps. Repeat 2 or 3 times to check for consistency, recording your results each try.

b. Walk the distance again, timing yourself with a stopwatch, to the nearest second. Repeat to check for consistency.

c. Make a ratio table that has 3 rows: one for distance in meters, one for number of steps, and one for seconds. Fill in the amounts in each row for each whole number of steps from 0 to 10.

d. Make another ratio table, counting in multiples of 10 steps from 0 to 100.

e. Make two sets of parallel number lines, one for part a and one for part b. Use graph paper, and mark scales in a way so that it's easy to look up various distances, numbers of steps, and times.

f. Check your measurements:

i. Find a building that has floor tiles 1 foot on a side. Note your beginning point. Take 20 steps in the same way you measured your walking. Use your number line to figure out how many feet that should be. Check by counting floor tiles.

ii. From the same starting point, walk for 10 seconds in the same way you measured your walking. Use your number line to figure out how many feet that should be. Check by counting floor tiles.

g. Answer using your number lines. If possible, check by doing it.

i. How long would it take you to walk a kilometer?

ii. If you walked for 45 minutes, how many kilometers would you go?

h. Decide on ratios that you can remember and use for mental calculations: one relating your steps and meters, and one relating seconds and meters. Practice walking and either counting steps or timing yourself. Use your remembered ratios to mentally calculate how far you went.

14. Weigh a car. Use tire gauges to measure the pressure in each of the tires, and find the area that each tire covers on the ground. (Tape paper or card stock around the tire, then measure the hole.

F. References and resources

Units, for future reference:

The distance around a running track, on the inside lane, is 400 meters.
The length of a football field is 100 yards.
The width of a football field is 50 yards.
1 mile = 1.6 kilometers
1 kilometer = 1000 meters
1 mile = 1760 yards = 5280 feet
1 minute = 60 seconds
1 hour = 60 minutes

Susan Lamon, *Teaching Fractions And Ratios For Understanding*, 2nd edition; Lawrence Erlbaum Associates, 2005. A very readable book designed for middle grades teachers. Includes many examples of student work and worked-out problems.