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Intervals and the Origins of Calculus

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Abstract. Interval (two-sided) estimates developed by John Wallis in the 17 century can be considered as an important step towards development of Newton-Leibniz's calculus.

1. Before Wallis

Greek mathematics left many problems unsolved, including the problem of finding the areas of even such simple geometric figures as a domain bounded by a hyperbola $y = 1 / x$ (i.e., $y = x^{-1}$). One of the first breakthroughs came around the year 1000, when the Arabic mathematician Abu Ali al-Hassan bel al-Hassan ben Haitam, or shortly al-Hassan (965–1038), known in the West as Alhazen, found the formulas for the area of a domain bounded by the curve $y = x^k$ (in modern terms, $\int x^k\,dx$) for an arbitrary nonnegative integer k [4], [5]. This discovery enabled Alhazen to compute the areas and volumes of the curves and surfaces bounded by polynomial equations $y = P(x)$.

The resulting formulas helped in solving of new important practical problems: Alhazen used these formulas to solve several practical problems of optics and astronomy. In the 14-th century, another practical reason appeared for computing areas: Oresme showed that if a curve represents the magnitude of velocity over time, then the area under the curve represents the total change in position ([3], p. 224). This idea has been extensively used by Galileo in the early seventeentha century.

Not all curves can be represented as $y = P(x)$ for a polynomial P. The simplest of such non-polynomial curves is a circle $x^2 + y^2 = 1$ (in functional terms, $y = \sqrt{1 - x^2}$). For the circle, the formula of the area was well known to the Greeks.

How can the circle be naturally embedded in a family of curves? This question lead John Wallis (1616–1703), a brilliant British theologician, mathematician, and linguist, to investigate the family of curves that are defined by 2-parametric equations $x^{1/p} + y^{1/q} = 1$ (or, in functional terms, $y = (1 - \sqrt[q]{x})^p$) for arbitrary values of parameters p and q. When p and q are not integers, computation of the area of this curve (i.e., of the integral $\int_0^1 (1 - \sqrt[q]{x})^p \, dx$) could not be easily done by the existing techniques. Therefore, a new technique had to be invented. This new technique (that later on led to what we now know as calculus) was discovered by John Wallis.

2. Enter Wallis and His Interval Estimates

In addition to theology* and linguistics**, Wallis *loved* to solve practical problems; he was very good in solving them. For example, in 1642, when Oliver Cromwell asked his help in deciphering the captured Royalist secret messages, he gladly took up the challenge and cracked the royal codes. Some historians believe that this success was crucial for the victory of the Parliamentary Party ([5], p. 6).

At the time when many mathematicians were more interested in the philosophical aspects of mathematical problems, Wallis was not initially interested in mathematics. He hardly learned anything of mathematics at school. His interest to mathematics started in about 1645, when he realized that the mathematical and physical problems of his time could be approached as if they were nature's codes to crack. For him, the natural approach to solving a problem was to lay down the cases for which, so to say, the code has been already cracked, and to try to guess the general pattern. With this method in mind, Wallis approached the problem of computing (in modern terms) the integral $\int (1 - \sqrt[q]{x})^p \, dx$. He did crack this code, and by 1655, he published his famous book *Arithmetica Infinitorum* in which he described the solution of this and of many other problems (reprinted in [8]).

How did he do it? The desired integral is always ≤ 1, therefore, to make analysis easier, Wallis used the reciprocal‡

$$f(p, q) = \frac{1}{\int_0^1 (1 - x^{1/p})^q \, dx}.$$

His idea was to lay down the cases for which the values of $f(p, q)$ was known, and try to interpolate. (Wallis not only invented the process of interpolation, he even invented the very word "interpolation".) For integer p and q, we can get the explicit formulas from Alhazen's results [4]:

* Wallis' discourses on Trinity are quoted in the histories of opinions on that subject.

** In 1653, Wallis wrote an English grammar for foreigners [7].

‡ We will, of course, reformulate Wallis' computations in modern notations (borrowed from [5]), for the convenience of modern readers. However, the tables are taken directly from Wallis' text.

q/p	0	1/2	1	3/2	2	5/2	3
0	1		1		1		1
1/2							
1	1		2		3		4
3/2							
2	1		3		6		10
5/2							
3	1		4		10		20

(A good exercise for the readers: without looking into the following text, try to fill in the remaining elements.)

For integer q, elements of the q-th row are described by a known (Alhazen's) polynomial of q-th order; we can use these polynomial formulas to fill these rows. Since the matrix is symmetric (because the curve $x^{1/p} + y^{1/q} = 1$ is clearly symmetric), we can thus fill the elements that correspond to integer p. The result of this interpolation is:

q/p	0	1/2	1	3/2	2	5/2	3
0	1	1	1	1	1	1	1
1/2	1		3/2		15/8		105/48
1	1	3/2	2	5/2	3	7/2	4
3/2	1		5/2		35/8		315/48
2	1	15/8	3	35/8	6	63/8	10
5/2	1		7/2		63/8		693/48
3	1	105/48	4	315/48	10	693/48	20

Wallis also noticed that for all p and q from this table, $f(p, q+1) = f(p,q) \cdot (p+q+1)/(q+1)$, and he guessed that the same formula must be true for all p and q. As a result, if we know $f(1/2, 1/2)$ (he denoted it by $\square$), we can compute

$$f(1/2, 3/2) = f(3/2, 1/2) = \frac{1/2+3/2}{3/2} f(1/2, 1/2) = \frac{4}{3}\square,$$
$$f(1/2, 5/2) = \frac{1/2+5/2}{5/2} f(1/2, 3/2) - \frac{6}{5} \cdot f(1/2, 3/2) = \frac{6}{5} \cdot \frac{4}{3}\square = \frac{8}{5}\square,$$

etc. In this manner, we can describe all elements of the table in terms of the unknown $\square$.

q/p	0	1/2	1	3/2	2	5/2	3
0	1	1	1	1	1	1	1
1/2	1	$\square$	3/2	$(4/3)\cdot\square$	15/8	$(8/5)\cdot\square$	105/48
1	1	3/2	2	5/2	3	7/2	4
3/2	1	$(4/3)\cdot\square$	5/2	$(8/3)\cdot\square$	35/8	$(64/15)\cdot\square$	315/48
2	1	15/8	3	35/8	6	63/8	10
5/2	1	$(8/5)\cdot\square$	7/2	$(64/15)\cdot\square$	63/8	$(128/15)\cdot\square$	693/48
3	1	105/48	4	315/48	10	693/48	20

Since the value $f(1/2, 1/2)$ corresponds to the circle, its value is well known: it is $4/\pi$. From the purely mathematical viewpoint, the problem is completely solved (the code is cracked). However, Wallis is interested in the practical computational problem. So, for him, an expression like $(4/3)\cdot\square = 16/(3\pi)$ is not the ultimate solution; there is still a problem of actually computing such values. To solve this problem, Wallis noticed that when q increases, the value $f(1/2, q)$ increases (this fact easily follows from our integral definition of $f(p, 1)$, because the integrated expression decreases with q). Therefore, for every integer n, $f(1/2, n - 1/2) < f(1/2, n) < f(1/2, n + 1/2)$. From the above recursive formula for f, we know that

$$f(1/2, n) = 1\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\ldots\cdot\frac{2n+1}{2n},$$

$$f(1/2, n - 1/2) = \frac{\square}{2}\cdot\frac{2}{1}\cdot\frac{4}{3}\cdot\ldots\cdot\frac{2n}{2n-1}.$$

Hence, we can conclude that

$$\frac{\square}{2}\cdot\prod_{k=1}^{n}\frac{2k}{2k-1} < \prod_{k=1}^{n}\frac{2k+1}{2k} < \frac{\square}{2}\cdot\prod_{k=1}^{n+1}\frac{2k}{2k-1},$$

from which we can deduce that the value $2/\square$ $(= \pi/2)$ belongs to the following interval:

$$\prod_{k=1}^{n}\frac{(2k)^2}{(2k-1)(2k+1)} < \frac{2}{\square} < \left[\prod_{k=1}^{n}\frac{(2k)^2}{(2k-1)(2k+1)}\right]\cdot\frac{2n+2}{2n+1}.$$

The larger n, the narrower this interval.

From this interval, we can get the intervals for all desired values p and q.

Comment. To avoid a wrong impression, it should be mentioned that this expression is not very practical, because it is very slowly convergent: e.g., five billion terms would be necessary to compute π with ten correct digits.

3. Historical Comment: Interval (Two-Sided) Estimates before Wallis

Wallis' computation was, of course, not the first two-sided (interval) estimate in mathematics. There have been many two-sided estimates before, in particular, several two-sided estimates for π; some of these estimates use polygons inside and outside the unit circle to bound π.

Two-sided estimates were known to Archimedes (?287–212 B.C.E.*) who used them to estimate areas and volumes (see, e.g., [2], Ch. 8). In particular, he found two-sided bounds for π by considering regular n-sided polygons inscribed in and circumscribed around the unit circle. Then, by comparing the perimeters p_n and P_n of these polygons with the perimeter 2π of the unit circle, he concluded that $p_n / 2 < \pi < P_n / 2$. Archimedes showed that if we know the perimeters p_n and P_n for some n, then we can compute the perimeters for polygons with $2n$ sides as $P_{2n} = 2p_nP_n / (p_n + P_n)$ and $p_{2n} = \sqrt{p_nP_{2n}}$. Archimedes himself used his formula to go from hexagons ($n = 6 = 2 \cdot 3$) for which perimeters were already known, to polygons with $12 = 2 \cdot 2^2$, $24 = 3 \cdot 2^3$, $48 = 3 \cdot 2^4$, and $96 = 3 \cdot 2^5$ sides. By considering 96-sided polygons, he proved the famous formula

$$3\frac{10}{71} < \pi < 3\frac{10}{70}.$$

This result appeared as one of the three propositions in his treatise *On the Measurement of the Circle*, the treatise that during the medieval times, was one of the most popular of Archimedes' books. Archimedes did not go further in his computations, because his formulas involve taking square roots and are, therefore, reasonably difficult to compute by hand.

Apollonius of Perga (?262–?190 B.C.E.) is said to have calculated the better bound in a book called *Quick Delivery*, but this book was lost. Further computations were made only centuries later:

- In the third century of our era, Liu Hui from China used polygons with $3072 = 3 \cdot 2^{10}$ sides and got six correct decimal digits of π. Liu Hui used a formula $a_{n+1} < \pi < a_n + 2 \cdot (a_{n+1} - a_n)$, in which a_n denotes the area of an inscribed regular $3 \cdot 2^n$-sided polygon, to compute two-sided bounds for π. Lui Hui's computations were continued even further by Tsu Ch'ung-chih (430–501), who, aided by his son Tsu Cheng-chih, got an estimate $3.1415926 < \pi < 3.1415927$. This two-sided bound for π was the best for nearly a thousand years; it is therefore fitting that a landmark on the Moon is named after Tsu Ch'ung-chih.
- After Tsu Ch'ung-chih, the first mathematician to continue Archimedes' computations was Jamshid ibn Mas'ud al-Kashi (?– ≈1436) from Ulug Bek's observatory. Ulug Bek, a grandson and heir of the great empire-builder Tamerlan (Timur), was a great patron of science; in his capital city Samarkand (now in Uzbekistan), Ulugbek built an observatory (it is still a tourist site), and invited the best scientists, Al-Kashi included, to its staff. Al-Kashi's main contribution to computations was his idea to use decimal fractions instead of sexagesimal (based on 60) that were then in use. These fractions helped him in long calcu-

* B.C.E. is an abbreviation for Before the Common Era; it means the same as B.C. Similarly, C.E. is an abbreviation for Common Era, which means the same as A.D.

lations, and he computed two-sided bounds for π that gave 16 correct decimal digits of π.

- Unaware of al-Kashi's results, another person continued Archimedes' computations: François Viète (1540–1603), also known by his Latinized name Franciscus Vieta. Approximately 100 years before Wallis, he used the $3 \cdot 2^{17}$-sided polygons to prove that

$$3.141{,}592{,}653{,}5 < \pi < 3.141{,}592{,}653{,}7.$$

Viète also gave the first known analytical expression for π:

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2} \cdot \sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2} \cdot \sqrt{\frac{1}{2} + \frac{1}{2} \cdot \sqrt{\frac{1}{2}}}} \cdots$$

Viète was well known to his contemporaries, but not so much as a mathematician, but as a... code breaker; he was so good in breaking codes that Spain officially complained to the Pope that Viète must be in league with the devil. In mathematics, he was known for his plea to use decimal rather than sexagesimal fractions.

- Probably the most impressive pre-calculus two-sided bound for π was calculated by Ludolph van Ceulen (1540–1610). In 1596, he used $15 \cdot 2^{37}$-sided polygons to compute π with twenty correct digits; later, he used regular $4 \cdot 2^{60}$-sided polygons to get two-sided bounds leading to 35 correct digits of π. The resulting value of π was engraved on his tombstone, and even the number π itself was for some time called the *Ludolphine constant.*

Other mathematicians (mainly from India) discovered alternating series for π, which also readily yield two-sided bounds.

- The earliest known series were described by Aryabhata (around 500 C.E.).
- Talaculattura, in his *Tantra Sangraha* (around 1608 C.E.) gave several series for π, among them the alternating series

$$\pi = 3 + 4\left(\frac{1}{3^3 - 3} - \frac{1}{5^3 - 5} + \frac{1}{7^3 - 7} - \cdots\right).$$

So, what was so special about Wallis' bounds? The special was that Wallis's work served as an important step towards the development of calculus.

4. After Wallis: Calculus

The most famous reader of Wallis was none else than the young Isaac Newton. In 1661, Newton studied his way through the book, ran several interpolation formulas of his own, and ended up with the infinite-series binomial formulas that lay the foundation of Newton's discovery of modern calculus.*

* It should be mentioned that calculus was independently discovered by Newton and Leibniz.

Thus, Wallis' work can be viewed as an important step towards development of calculus.

5. After Calculus: Wallis

And what did Wallis do after Newton's famous works appeared? He did not do any more original math. Instead, he used his "code-cracking" skills to solve serious real-life problems. In particular, he worked for the project of draining the English fens, i.e., to create the farmland out of salt marshes: he tried (and succeeded) in finding the patterns in the dynamics of water level.

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