

Drawing Logarithmic Curves with *Geometer's Sketchpad*: A Method Inspired by Historical Sources

David Dennis

&

Jere Confrey

Mailing Address:

Dr. David Dennis

4249 Cedar Dr.

San Bernardino, CA 92407

Tel: 909-883-0848

e-mail: david.dennis@earthlink.net

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Abstract: We start by describing a mechanical linkage device from Descartes' *Geometry* which draws a series of curves which can locate an arbitrary number of geometric means between any two lengths. Using this linkage combined with Napier's original conception of logarithms as covariation between geometric and arithmetic sequences, one can then construct arbitrarily dense sets of points on any log curve. When this construction is created on *Geometer's Sketchpad* one obtains a family of curves with two physically adjustable parameters, which allow any geometric sequence to be mapped against any arithmetic sequence. The derivative of any logarithm is then obtained using only similar triangles, and other geometric properties are investigated, such as the constant subtangent of exponential curves. Related educational issues are discussed.

Introduction:

In this article we first describe a mechanical linkage device from Descartes' *Geometry* (1637), for finding any number of geometric means. This linkage device draws a family of curves which we will discuss briefly. We then employ Descartes' device as part of a construction that will find any number of points, as densely as desired, on any logarithmic or exponential curve. This second construction uses the dynamic geometry software, *Geometer's Sketchpad*, in a modern adaptation of several historical ideas, combined in ways that provide useful and enlightening curricular suggestions, including an elementary approach to tangent slopes. Several important educational and epistemological questions are raised by our approach, and we conclude by discussing these.

Descartes' Construction of Geometric Means:

In the *Geometry* (1637), Descartes considered the problem of finding n mean proportionals (i.e. geometric means) between any two lengths a and b (with $a < b$). To do this one must find a sequence of lengths beginning with a and ending with b such

that the ratio of any two consecutive terms is constant. In modern algebraic language, this means finding a sequence x_0, x_1, \dots, x_{n+1} such that for some fixed ratio r , $x_k = a \cdot r^k$ and $x_{n+1} = b$. Hence the terms of the sequence have a constant ratio of r , and form a geometric sequence beginning with a and ending with b .

Descartes began, as always, with a geometric construction. He imagined a series of rulers with square ends sliding along and pushing each other creating a series of similar right triangles (see Figure 1, which is reproduced from an original 1637 edition of the *Geometry*). Let Y be the origin with A and B on a circle of radius a centered at Y . The tangent to the circle at B intersects the line XY (the x -axis) at the point C . As angle XYZ increases, C moves further to the right on the x -axis. The vertical from C then intersects the line XY at D which is still further from the origin. The triangles YBC , YCD , YDE , YEF , etc. are all similar, since they are all right triangles that contain the angle XYZ . Hence we have $\frac{YB}{YC} = \frac{YC}{YD} = \frac{YD}{YE} = \frac{YE}{YF} = \dots$

Therefore the sequence of lengths $a=YB, YC, YD, YE, YF, \dots$ form a geometric sequence.

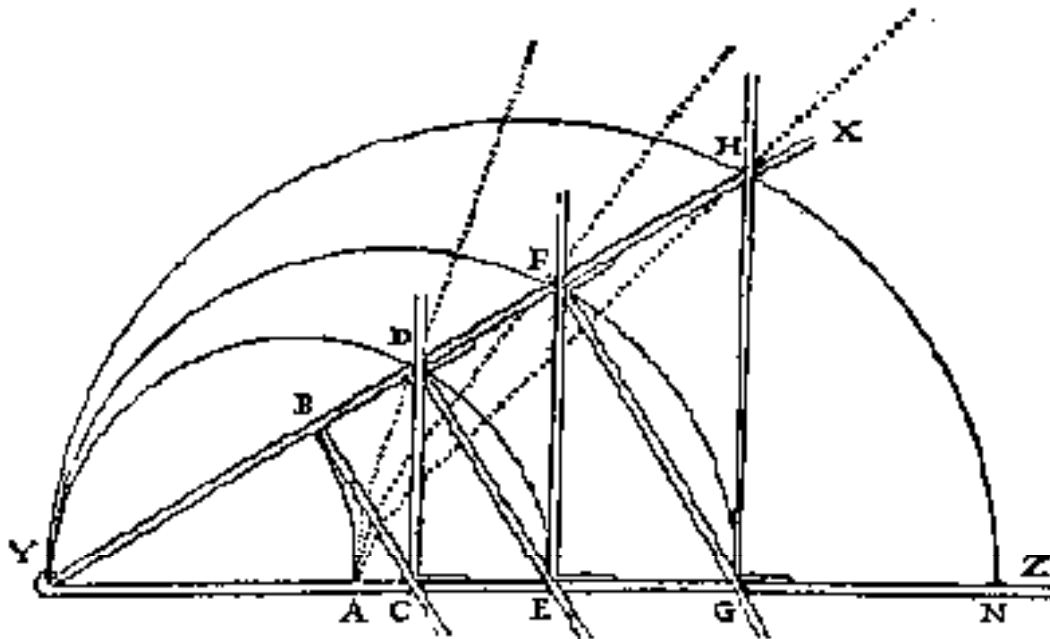


Figure 1

If we let $a=1$ and angle $XYZ=60^\circ$, we form the sequence: 1,2,4,8,16,32,.....

If angle $XYZ=45^\circ$, the sequence of lengths is: $1, \sqrt{2}, 2, 2\sqrt{2}, 4, 4\sqrt{2}, 8, \dots$ which is a refinement of the previous sequence. As angle XYZ decreases, we obtain increasingly dense geometric sequences. In modern terms the relationship between the constant ratio r and angle XYZ is given by the equation: $\sec(\text{angle } XYZ)=r$. This relationship is never mentioned in the *Geometry*. Descartes instead emphasized the curves traced by the points D, F, and H as the angle XYZ is varied (shown in Figure 1 by the dotted lines). These curves all have algebraic equations, as opposed to the secant which can only be computed with some infinite process.

To solve the original problem of finding n mean proportionals between a and b , Descartes suggested using the curves drawn by the device. For example, if two mean proportionals are sought between a and b , mark off length $b=YE$ on the line YZ (recall $a=YA$). Next construct the circle having diameter YE , and find its intersection (D) with the first of these curves (see Figure 1). Then drop the vertical line from that point (D) to YZ , to locate the point C. YC and YD will then be the desired mean proportionals. This method uses the curve drawn by D to determine the appropriate angle of the device so that the point E will fall on any specified length.

The equations of the curves traced by D, F, H, . . . can all be found by successively substituting into the similarity relations upon which the device was built. To find these equations one can proceed as follows. Let $Y=(0,0)$, $D=(x,y)$, and let $YD=z$. Now $z^2=x^2+y^2$, but one also knows that $\frac{z}{x} = \frac{x}{a}$. Hence $z = \frac{x^2}{a}$, and therefore by substitution one obtains, for the path of point D, the equation: $x^4 = a^2(x^2 + y^2)$.

Now let $F=(x,y)$, and let $YF=z$. Now $\frac{z}{x} = \frac{x}{YD}$, hence $YD = \frac{x^2}{z}$. One also knows that $\frac{x}{YD} = \frac{YD}{YC}$, so substituting and solving for YC one gets: $YC = \frac{x^3}{z^2}$. Lastly one

knows that $\frac{YD}{YC} = \frac{YC}{a}$, and hence: $\frac{ax^2}{z} = \frac{x^6}{z^4}$. Solving for z one obtains: $z = \sqrt[3]{\frac{x^4}{a}}$.

As before $z^2=x^2+y^2$, so $\sqrt[3]{\frac{x^8}{a^2}} = x^2 + y^2$. Cubing both sides one obtains, for the path of point F, the equation: $x^8 = a^2 (x^2 + y^2)^3$.

In a similar fashion, one can find that an equation of the curve traced by the point H is: $x^{12} = a^2 (x^2 + y^2)^5$. Note that all of these curves pass through the point $A=(a,0)$, and that as one moves from one of these curves to the next the degree of the equation always increases by four (on both sides of the equation). In the *Geometry*, Descartes proposed a system which classified curves according to pairs of algebraic degree, i.e. lines and conics form the first "genre" of curves; those with third and fourth degree equations form the second "genre" of curves; and so on.¹ In many examples, Descartes found that iterating various forms of mechanical linkages tended to jack up by twos the degrees of the equations of the curves. The curve traced by D is of his second *genre*; the curve traced by F is of the fourth *genre*; etc. To get from any one of these curves to the next one involves two perpendicular projections, each of which raises by one the Cartesian *genre* of the curve.

Descartes, after stating that, "there is, I believe, no easier method of finding any number of mean proportionals, nor one whose demonstration is clearer," (1637, p. 155) goes on to criticize his own construction for using curves of a higher *genre* than is necessary. Finding two mean proportionals, for example, is equivalent to solving a cubic equation and can be accomplished by using only conic sections (first *genre*) while the curve traced by D is of the second *genre*. The solution of cubics by intersecting conics had been achieved in the thirteenth century by Omar Khayyam and was well known in seventeenth century Europe (Joseph, 1991). Descartes spent much of the latter part of the *Geometry* discussing the issue of finding curves of minimal *genre* which will solve various geometry problems (1637).

¹ This same classification by pairs of degrees is used in modern topology in the definition of the "genus" of a surface. Classification by pairs of algebraic degrees often makes more sense geometrically.

Descartes expounded an epistemological theory which sought a universal structural science of measure which he called "mathesis universalis" (Lenoir, 1979). Fundamental to his program was his classification of curves in geometry. He wanted to expand the repertoire of curves that were allowed in geometry beyond the classical restrictions to lines and circles, but he only wanted to include curves whose construction he considered to be "clear and distinct" (Descartes, 1637). For him this meant curves which could be drawn by linkages (i.e. devices employing only hinged rods and pivots). Such curves could all be classified by his system according to pairs of algebraic degrees, since the class of curves that can be drawn by linkages is exactly those which have algebraic equations (Artobolevski, 1964). Algebraic curves were called "geometrical" by Descartes because he wanted to expand the constructions allowed in geometry to include those curves. All other curves he called "mechanical."

This distinction is equivalent to what Leibniz would later call "algebraic" and "transcendental" curves. Descartes viewed "mechanical" (i.e. transcendental) curves as involving some combination of incommensurable actions. Examples that he specifically mentioned are the spiral, quadratrix, and cycloid. These curves all involve a combination of rotation and linear motion that cannot be connected and regulated by some linkage. The drawing of such curves involves rolling a wheel or the unwinding of string from a circle. Descartes was aware that such curves could not be classified by his system. This is not to say that Descartes did not address himself to problems concerning transcendental curves (see Dennis, 1995, for some of his thoughts on the cycloid).

A Dynamic Construction of Logarithmic Curves:

In order to construct logarithmic curves, we must first define the term "subtangent." Given a smooth curve and an axis line, for each point P on the curve let T be the intersection of the tangent line to the curve at P and the axis, and let O be the foot of the perpendicular from P to the axis (i.e. PO is the ordinate, see Figure 2). For

each point P the subtangent is then defined as the line segment TO. Throughout the seventeenth century, such geometric entities were studied for symmetry and invariance as the point P moved along the curve. Such investigations played a very important role both in the study of curves and in the development of the concept of functions (Arnol'd, 1990; Dennis & Confrey, 1995).

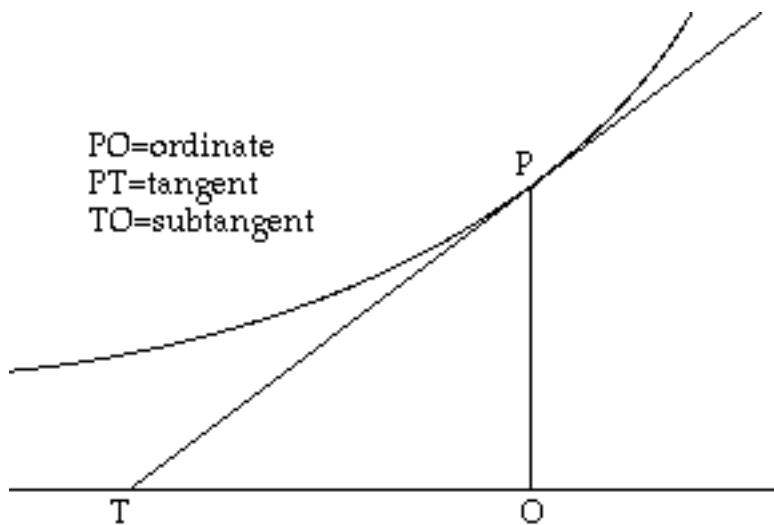


Figure 2

Two years after the publication of the *Geometry* Descartes addressed a problem that was sent to him by De Beaune which asked for the construction of a curve in a skewed coordinate system where the ratios of the subtangents to the ordinates are everywhere equal to the ratio of the ordinates to a fixed segment, i.e. a type of logarithmic curve, the requirement being equivalent to a first order differential equation (Lenoir, 1979). Descartes generated a method for pointwise approximation of this curve and also provided a detailed study of how the curve could be drawn by a combination of motions with particular progressions of speeds. He then stated:

I suspect that these two movements are incommensurable to such an extent that it will never be possible for one to regulate the other exactly, and thus this curve is one of those which I excluded from my *Geometry* as being mechanical; hence I am not surprised that I

have not been able to solve the problem in any way other than I have given here, for it is not a geometrical line. (Descartes, quoted in Lenoir, 1979, p. 362)²

We will now proceed to construct pointwise approximations of logarithmic curves, but we will not follow the particular example discussed in Descartes' letters to De Beaune. That example turns out to have been a transformation of a logarithm added to a linear function. We will instead construct standard logarithmic curves using Descartes' device shown in Figure 1, together with the original conception of logarithms by Napier as pairings of geometric and arithmetic sequences (Smith & Confrey, 1994; Edwards, 1979).

This construction connects curve drawing with a covariational view of functions. Covariation is essentially a viewing of tables of data that looks for methods to simultaneously extend or interpolate values in both columns separately, rather than looking for a rule which relates values in the first column to those in the second. This approach to functions was central in the thinking of Leibniz (Leibniz, 1712) and has been shown to be important in the thinking of students (Rizzuti, 1991; Confrey & Smith, 1995).

Napier and others in the early seventeenth century made tables of logarithms by placing arithmetic sequence alongside geometric sequences. They devised various ways to make these tables dense (Edwards, 1979). These early approaches to logarithms were entirely tabular and calculational and did not involve curves or graphs. When Descartes constructed a curve as a solution to De Beaune's problem he did not view the curve as a "logarithm." A fully flexible view that could go back and forth between curves, tables and equations did not evolve until the end of the

² For a fascinating social and philosophical analysis of why Descartes would adopt such an attitude see the article by Lenoir (1979). It certainly had nothing to do with his ability to contend with such problems.

seventeenth century, especially with respect to transcendental curves whose general coordinates could only be found by using series expansions (Dennis & Confrey, 1993).

Our aim here is to provide modern students with a hands-on way to build logarithmic and exponential curves through a series of simple geometric constructions using *Geometer's Sketchpad*. From the standpoint of covariation there is little difference between exponentials and logs. The pair of actions which builds one also builds the other. We have constructed the following curves as logarithms, but the same constructions could be viewed as exponentials, by simply repositioning the constructive actions.

We start by building a computer simulation of Descartes' device for the construction of geometric sequences with $a=1$ (see Figure 3). Let O be the origin and let H be any point on the unit circle. By moving H around the circle the distances of the labeled points from the origin will form geometric sequences with any common ratio. That is if $r = OG_1$, then $r^2 = OH_2$, $r^3 = OG_3$, $r^4 = OH_4$, etc.

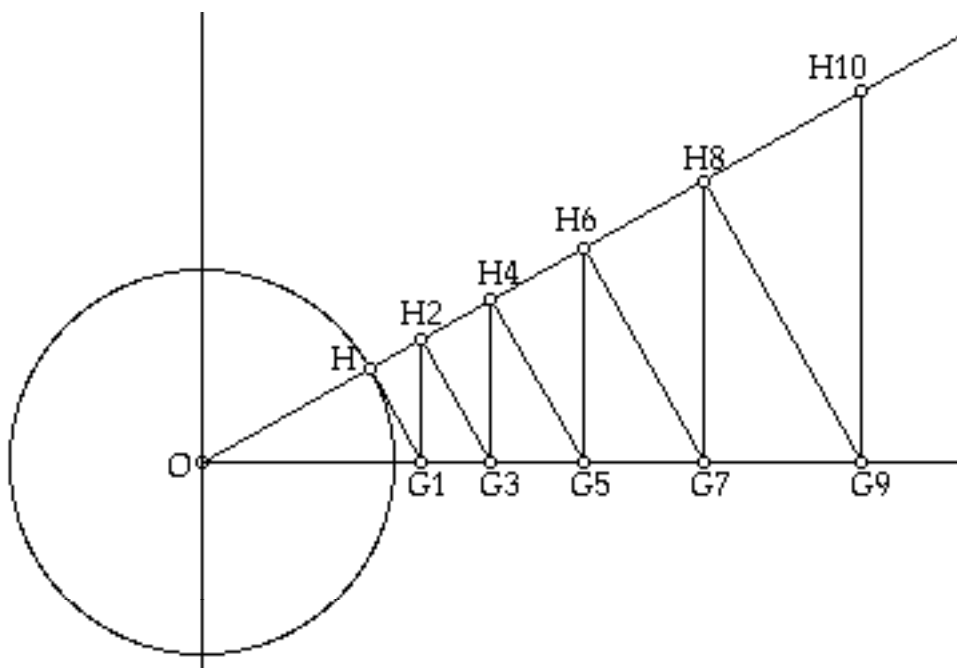


Figure 3

This construction can also be extended to the interior of the unit circle to obtain segments whose lengths are the negative powers of r . Once again, as with the preceding construction, the odd powers of r are on the horizontal while the even powers of r are on the line OH. This can be seen, as before, by considering the series of similar triangles with common vertex O (see Figure 4).

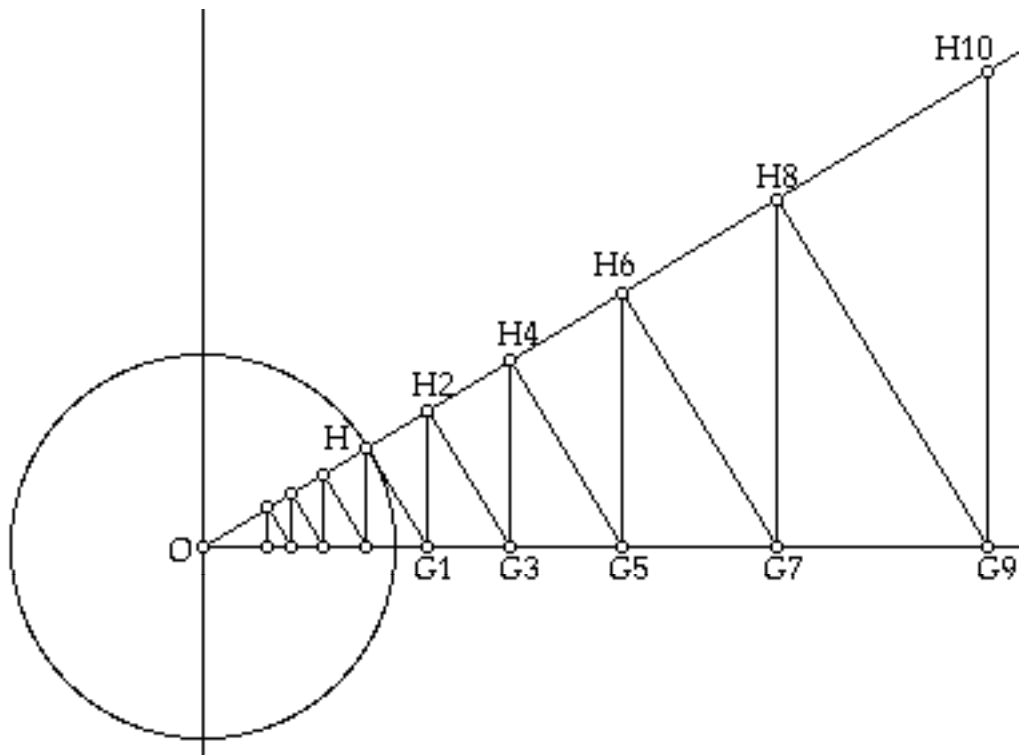


Figure 4

In order to get the entire geometric sequence on one line, we will now transfer the lengths marked on the line OH onto the x -axis by using circles centered at O (see Figure 5). Thus we now have a geometric sequence laid out on the x -axis whose common ratio, or density, can be varied as the point H is rotated. The point where the circle through H_2 intersects the x -axis, we will name G_2 , likewise for H_4 , H_6 , etc. The points on the x -axis that are inside the unit circle we will call G_{-1} , G_{-2} , \dots etc. where the subscripts correspond to the powers of r that represent their distances from O.

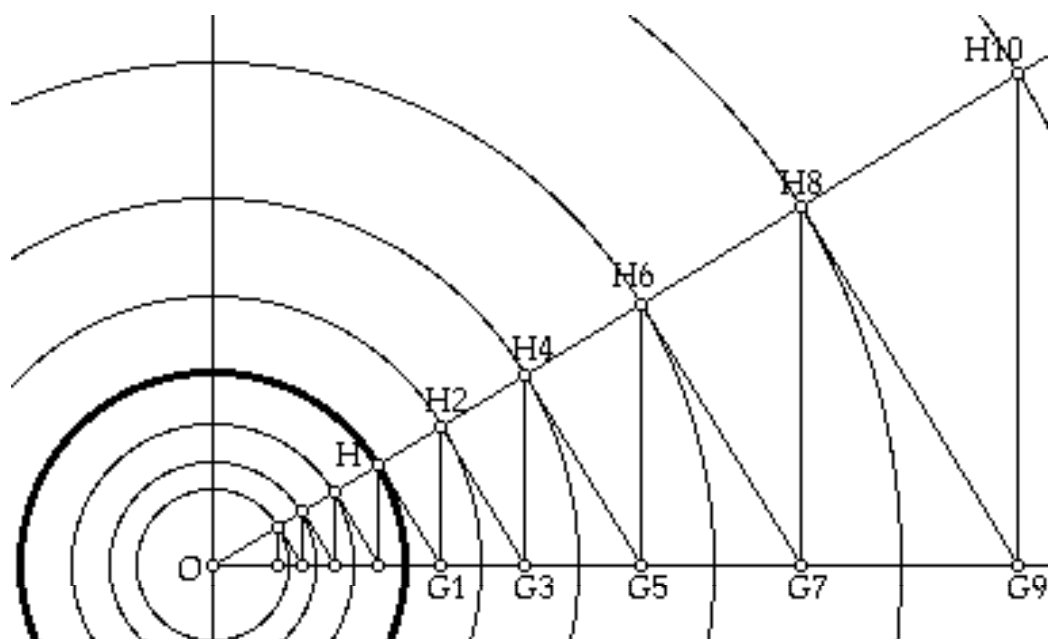


Figure 5

In order to construct logarithmic curves we must now construct an arithmetic sequence $\{A_i\}$ on the y -axis with a variable common difference. This can be achieved in a variety of ways (e.g. create OA_1 , and then translate repeatedly along the y -axis by OA_1). The common difference (d) in the arithmetic sequence can be adjusted here by moving A_1 along the y -axis. We next vertically translate each of the points G_i in the geometric sequence by lengths OA_i corresponding to consecutive points in the arithmetic sequence, thus creating the points (G_i, A_i) . These translated points will then lie on the graph of a logarithm. We can connect the new points by line segments to approximate a log curve (see Figure 6). Using the measuring tools available in *Geometer's Sketchpad*, one can continuously monitor the lengths in both the sequences, and hence the coordinates of the points on the log curve. Some of the construction lines in Figure 6 have been hidden for greater visual clarity.

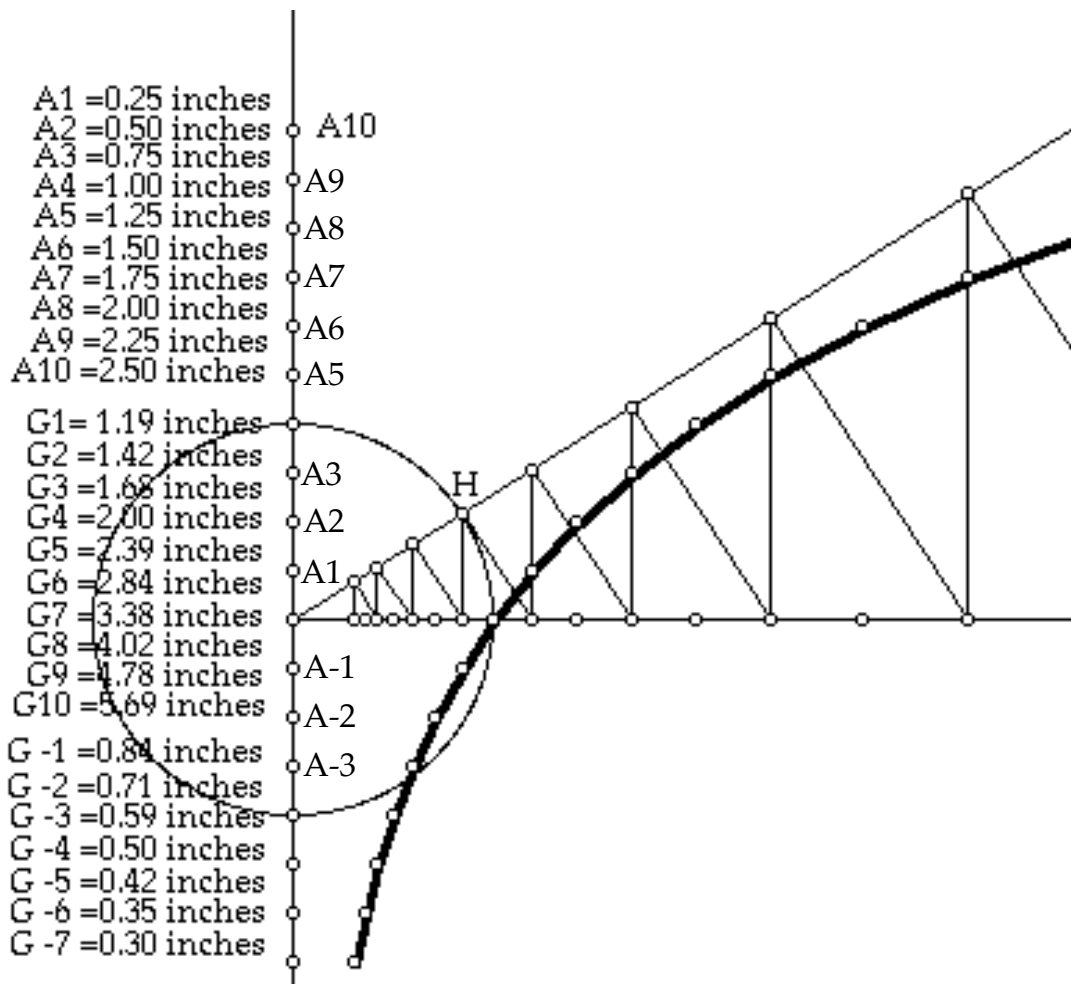


Figure 6

We now have an adjustable curve. By moving H around the unit circle or A₁ along the y-axis, one can map any geometric sequence against any arithmetic sequence. In Figure 6 the point H is adjusted so that G₄=2, and the point A₁ is adjusted so that A₄=1. Hence this curve is a graph of the log base 2.

By readjusting A_1 so that $A_8=1$ the curve shifts dynamically to become a graph of the log base 4 (see Figure 7).

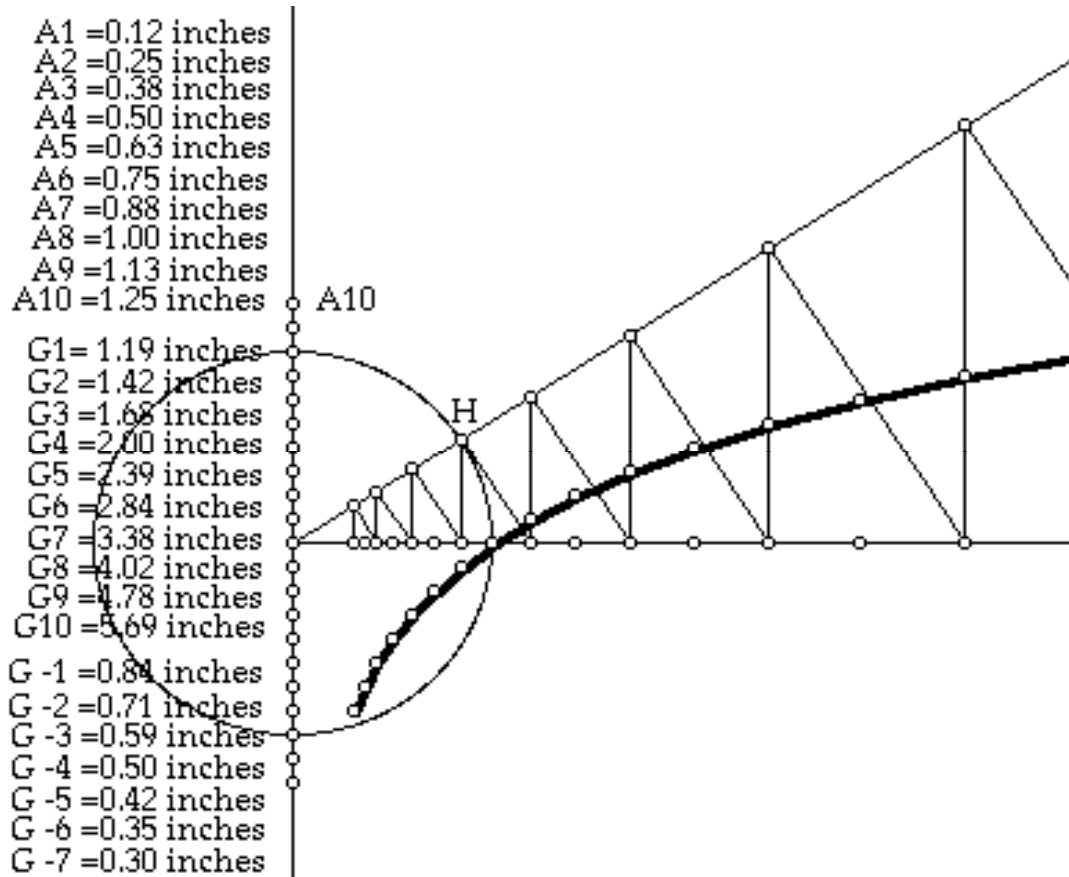


Figure 7

By readjusting the point H so that $G_8=3$, one obtains a graph of the log base 3 (see Figure 8). Since the monitors are measuring to the hundredth of an inch, which is smaller than a pixel, it is not always possible to get exactly the desired numbers by simple direct actions (e.g. G_8 reads 2.96 in Fig. 7). One can get around the pixel problem by using the appropriate rescaling window commands, but for a first experience this would decrease the sense of a direct physical approach, which we feel is important for students.

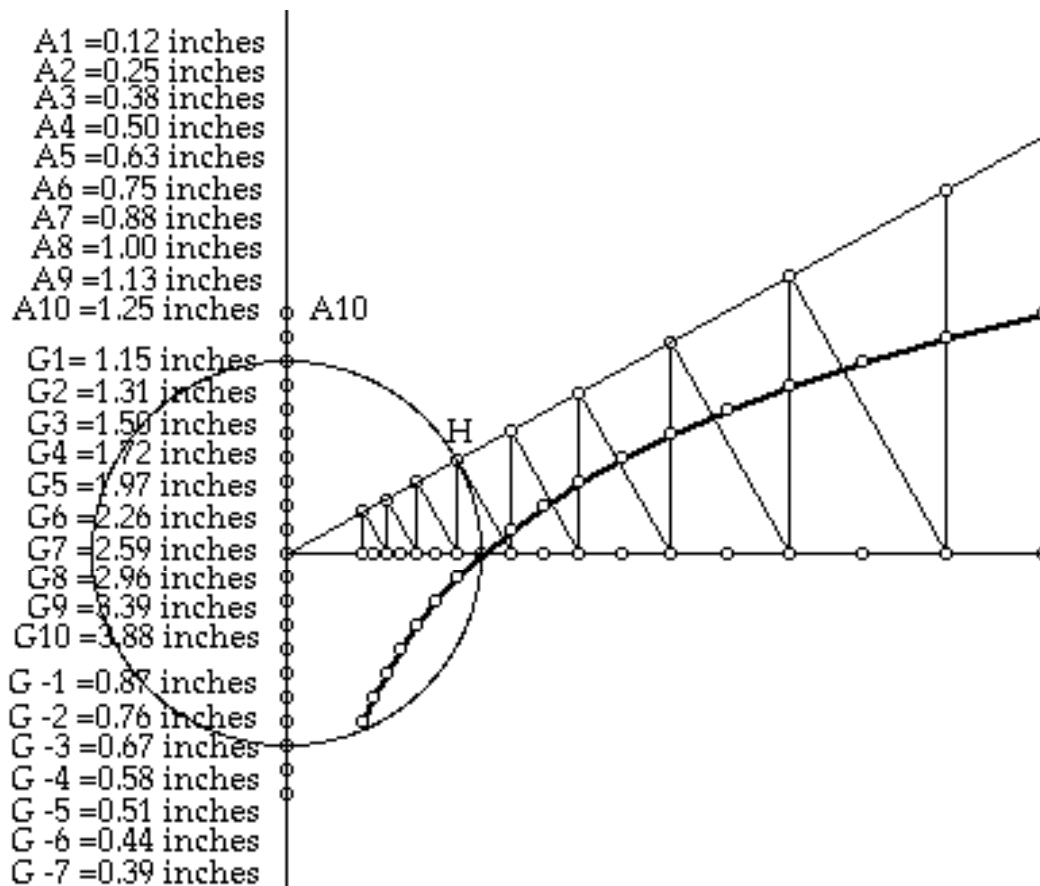


Figure 8

It is fascinating to watch these curves flex and bend as the arithmetic and geometric sequences are manipulated. Even when the points are quite broadly spaced, as in Figure 6, the graphs look very smoothly curved, though they are all made up of line segments. When the angle of H is increased, the geometric sequence spreads out rapidly off of the screen. By scanning far to the right, it is instructive to see just how incredibly flat log curves become.

When the arithmetic and geometric sequences are both spread out, the graphs can eventually become "chunky" since the points are being connected with line segments. However, by manipulating both sequences it is possible to increase the density of points on any particular log graph without changing the base. For example, we could create another graph of the log base 2 by setting $A_8=1$, and $G_8=2$ (see Figure 9). This is the same curve as the one in Figure 6, but with a much higher density of

constructed points. Descartes' device allows us to geometrically carry out the calculational aims of Napier and other seventeenth century table makers. Geometric sequences can be built as densely as one desires, and paired against any arithmetic sequence.

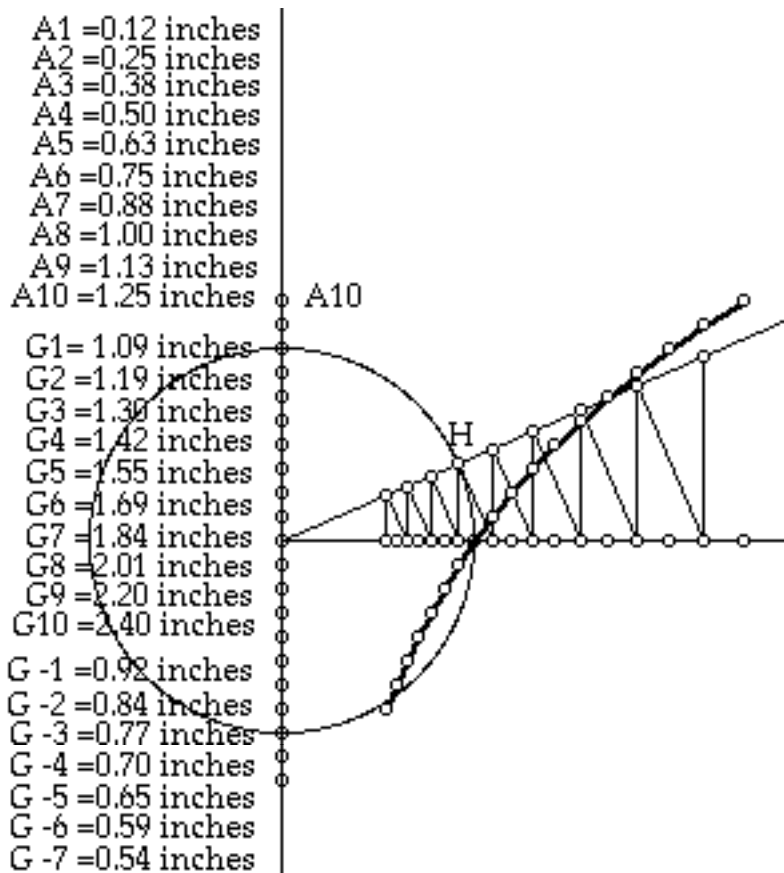


Figure 9

An Investigation of the Slopes of Logarithmic Curves:

After looking at these log curves shift and bend dynamically, one can begin to look carefully at the slopes of segments that join points on the curves. Several interesting patterns come to light. If the slopes of segments that join consecutive constructed points are used to approximate the tangent slope at a point, say for example at (1,0), it is visually apparent that this calculation is not the best thing to use. The slope between the nearest points to the right and left of a point gives a better approximation of the tangent slope at that point. This is true for most curves, not just

the logarithm.³ Here, at (1,0), the best approach to the tangent slope is to calculate the secant slope between G_{-1} , and G_1 . Letting r equal the common ratio of the geometric sequence, and d equal the common difference of the arithmetic sequence, we calculate this slope as:

$$\text{tangent slope at } (1,0) \approx \frac{2d}{r - \frac{1}{r}} = \frac{2rd}{r^2 - 1}$$

We will use k for this approximate slope at (1,0). Suppose we now approximate in the same way the slope at any other point on the constructed curve. The approximate tangent slope at $(G_n, A_n) = (r^n, nd)$ is found by computing the secant slope between G_{n-1} , and G_{n+1} . The calculation yields:

$$\text{tangent slope at } (G_n, A_n) \approx \frac{2d}{r^{n+1} - r^{n-1}} = \frac{1}{r^n} \cdot \frac{2rd}{r^2 - 1} = \frac{k}{r^n}$$

Here one has the approximate tangent slope at a point on a logarithmic curve written as the function $\frac{1}{x}$ times a constant k that is the slope of the curve at (1,0). Of course these slopes are all approximations, but once the slope at (1,0) is approximated it can be divided by the x -coordinate at any other point to get the corresponding slope approximation at that point. By making the constructed points on the curve denser, the approximations all improve by the same factor. Thus the essential derivative property of logarithms is revealed without recourse to the usual formalisms of calculus. In fact, even more is being displayed here than the usual derivative of a logarithm. One sees that the all the slope approximations converge uniformly, as the density of the constructed points is increased.

³ It is strange that when the derivative is developed in calculus classes, it is defined using secant slopes from the point in question, rather than around the point. It would seem that nobody is directly interested in secant slope approximations, except as an algebraic device from which to define a limit. The practical geometry of using secant slopes is ignored.

This constant k (=the slope at $(1,0)$) can be seen geometrically in another way. If we view these curves and tangent constructions using the vertical axis (i.e. as exponential functions), then we find that the subtangent is constant for all points along the curve, and is always equal to k . This can be established algebraically from the previous discussion, but it is nice to see it geometrically on the curve, and verify it using the measurement capability of *Geometer's Sketchpad*. This is shown in Figure 10 for two different points on the log base three curve. The tangent lines and slope are approximated by using the points adjacent to the one under consideration and the accuracy is quite good (a calculator gives $k=.910$).

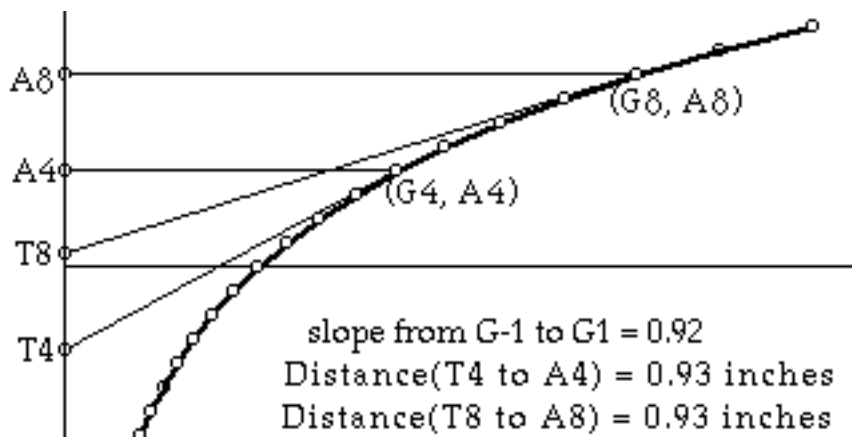


Figure 10

This constant subtangent property was at the heart of Descartes' discussion of De Beaune's curve. The constant subtangent was the hallmark by which logarithmic and exponential curves were recognized during the seventeenth century (Lenoir, 1979; Arnol'd 1990). One way to think of this property is to imagine using Newton's method to search for a root of an exponential curve. The method will march off to infinity at a constant arithmetical rate, where the size of the steps will be the constant k .

In order to construct the natural logarithm, we want the slope k at $(1,0)$ to be equal to 1. This is the property from which Euler first derived the number "e" (Euler, 1748). Returning to the construction, with a measure that monitors the secant slope k between G_{-1} , and G_1 , we now rotate H until the slope measurement reads as close to 1

as possible. We have now constructed a close approximation to the graph of the natural logarithm. The approximate slope at any point on the curve is the inverse of its x-coordinate. Note that since $A_5 = 1$, the value of G_5 is approximately the number "e" (see Figure 11).

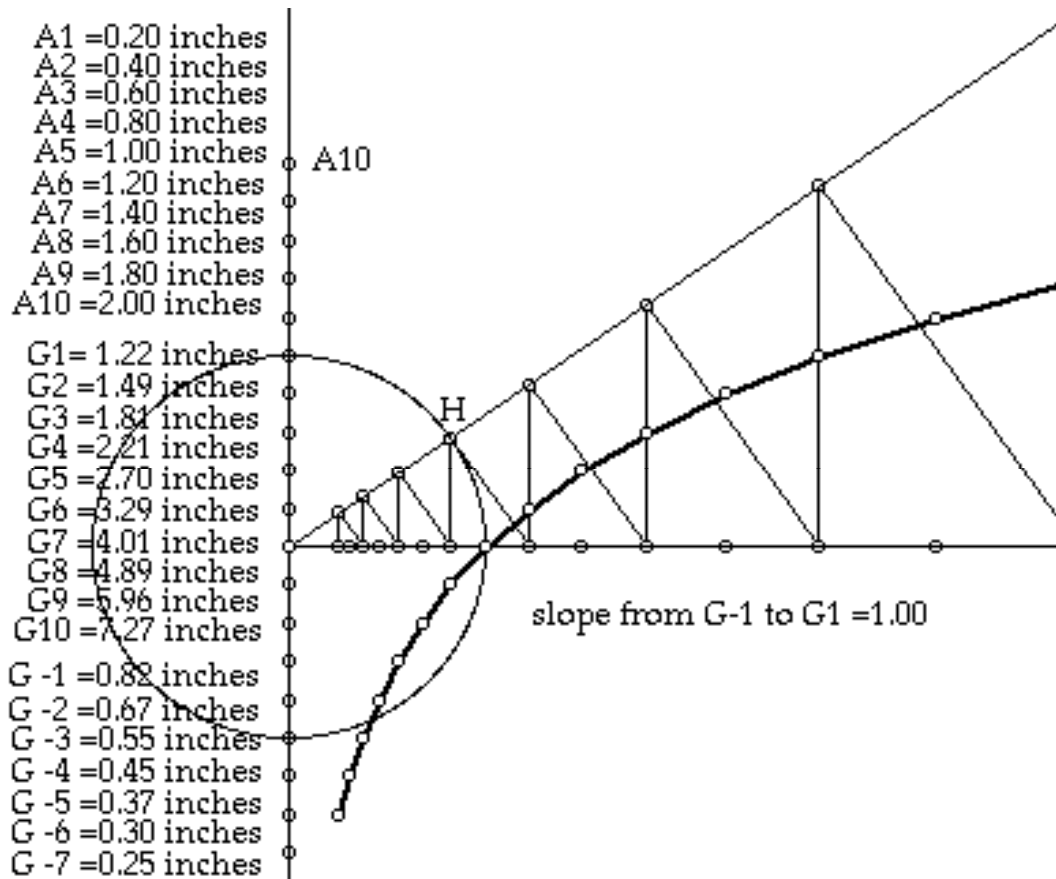


Figure 11

This geometric construction of points on log curves achieves the goals set out by Napier. It allows one to construct logarithms (and also exponents) as densely as one desires. Of course Napier achieved these goals through interpolation schemes (Edwards, 1979), and throughout the seventeenth century increasingly subtle methods of table interpolation were developed, e.g. those of John Wallis (Dennis & Confrey, 1996). The story of these calculational techniques is a very important one leading eventually to Newton's development of binomial expansions for fractional powers. Euler routinely used Newton's binomial expansion techniques to calculate log tables to

over 20 decimal places (Euler, 1748). Theoretically the geometric construction has unlimited accuracy, and this can be achieved through appropriate rescalings, but directly using measuring capability of *Geometer's Sketchpad*, one is limited to at most three decimal places.

Conclusions:

Our approach raises several general educational and epistemological questions:

- 1) How are tools and their actions related to mathematical language and symbols?
- 2) What notion of functions arises from constructions made with particular tools?
- 3) What are the educational and epistemic roles of mathematical history?

The first question is addressed in theoretical detail by Confrey (1993). She suggests that effective education in mathematics must be approached as a balanced dialogue between "grounded activity" and "systematic inquiry." New technological tools, such as dynamic geometry software, help to populate the dialogue between physical investigations and symbolic language, allowing it to flow with greater ease. Confrey stresses the need for balance in this dialogue. All too often in our mathematics curriculum, symbolic language is given a preeminent role, de-emphasizing visual and physical activity, especially at more advanced levels. The examples discussed in this paper demonstrate the value of striking a balance.

The most common notion of a function that is taught in mathematics is to see it as a rule or process which computes or predicts one quantity from another, i.e. a correspondence notion. These quantities are most often real numbers, and functions can then be represented by graphing these quantities in the plane. Our investigations employed two different conceptions of functions, which have both played important historical roles. The first conception involved creating a curve as a primary object through some kind of physical or geometric action, and only afterwards analyzing it by means of the geometric properties of particular magnitudes, such as abscissas, ordinates, and subtangents (Dennis, 1995). Algebra and equations then become

secondary representations of curves, rather than primary generators. The second conception of functions that came into play in our constructions is "covariation" (Confrey & Smith, 1995). This view sees functions as a pair of independent operations on columns in a table, where one finds ways to simultaneously extend or interpolate values on both sides of a table thus generating a correspondence when the columns are paired in a given position.

Both of these conceptions of functions have played important roles in the genesis of analytic geometry and calculus. Neither one is entirely reducible to the current "definition" of a function. For example, in Descartes' *Geometry*, never once is a curve created by plotting points from an equation. Descartes always began with a physical or geometric way of generating a curve, and then by analyzing the actions which produced the curve, he obtained equations. Curves were primary and gave rise to equations only as a secondary representation. Equations allowed him to create a taxonomy of curves (Lenoir, 1979). Furthermore, Leibniz originally created his calculus notation from his many experiments with tables, where a covariational approach was his fundamental form of generation (Leibniz, 1712).

These alternative conceptions of functions are not just awkward phases of historical development which should be abandoned in light of more modern developments. Quite the contrary, they provide ways of working which give mathematicians a powerful flexibility. They can help to balance the dialogue between the physical world and our attempts to represent it symbolically, especially when combined with newly available tools, such as *Geometer's Sketchpad*.

We ask the reader to compare this investigation of logarithms with the approaches more frequently taken in classrooms. Many students are introduced to logarithms in a formal algebraic way, with no references to geometry or to table construction. Such students often have no method for geometrically or numerically constructing, even a square root. Such an approach leads, at best, only to a superficial

understanding of the grammar of logarithmic notation. There is no dialogue at all between geometrical, numerical and algebraic experience.

Another approach that is frequently taken is to see logarithms as the accumulated area under a hyperbola (usually $y=1/x$). This approach can provide many fascinating insights that connect logarithms to both geometry and to the numerical construction of tables. The study of hyperbolic area accumulation was fundamental in the early work of Newton, but was always linked in his work to extensions of the table interpolations of John Wallis. It was in this setting that Newton created his first infinite binomial expansions (Dennis & Confrey, 1993; Edwards, 1979). Although the hyperbolic area approach can create a fascinating and balanced dialogue, it is not usually taken with students until they are already involved with calculus. The fundamental theorem of calculus, for example, is usually invoked to show that the hyperbolic area function must have a derivative of $1/x$.

The approach that we have described here is strictly pre-calculus. It involves only a systematic use of similar triangles, in a hands-on setting that is both visual, physical, and geometric. It provides a specific form of grounded activity that allows students to manipulate, extend, and interpolate both logarithms and continuous exponents. Rather than using calculus to create a balanced dialogue, this approach uses the dialogue to achieve some of the results of calculus in a very simple setting. It highlights the power of iterated geometric similarity (Confrey, 1994).

Reading this paper cannot truly convey the feeling one gets while physically manipulating the curves. The investigation of the slopes of log curves depends logically only on the properties of a table which maps a geometric sequence against an arithmetic sequence, but we did not notice this piece of algebra until many fluctuating examples of log curves had appeared on the screen. The geometry can heighten the intuition so that fruitful conjectures emerge. The power of suggestion should not be underestimated. The association of rotation around the unit circle with the building of

logarithms is a wonderful foreshadowing of the connections between these functions and the trigonometric functions, when extended to the complex numbers (Euler, 1748).

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